Carleman Estimates of Refined Stochastic Beam Equations and Applications∗

Yongyi Yu† and Ji-Feng Zhang‡

Abstract

This paper is devoted to establishing global Carleman estimates for refined stochastic beam equations. First, by establishing a fundamental weighted identity, two Carleman estimates are derived with different weight functions for the refined stochastic beam equation, which is a coupled system consisting of a stochastic ordinary differential equation and a stochastic partial differential equation. As applications of these Carleman estimates, the exact controllability of the refined system is proved by the least controls in some sense. Different from classical stochastic beam equations, the refined one is exactly controllable at any time. Meanwhile, the uniqueness of an inverse source problem for refined stochastic beam equations is obtained without any requirement on the initial and terminal values.

AMS subject classifications. 93B05, 93B07

Key Words. refined stochastic beam equation, Carleman estimate, exact controllability, inverse problem

1 Introduction

Let $T > 0$, $Q = (0, 1) \times (0, T)$ and $G_0$ be a nonempty open subset of $(0, 1)$. Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined such that $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathcal{H}$ be a Banach space, and let $L^2_{\mathcal{F}}(0, T; \mathcal{H})$ be the Banach space consisting of all $\mathcal{H}$-valued $\mathcal{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; \mathcal{H})}) < \infty$; $L^\infty_{\mathcal{F}}(0, T; \mathcal{H})$ denotes the Banach space consisting of all $\mathcal{H}$-valued

∗This work was supported by National Key R&D Program of China under Grant 2018YFA0703800, National Natural Science Foundation of China under Grant 61877057, and Postdoctoral Science Foundation of China under Grants 2021TQ0353.

†Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. E-mail address: yuyy122@amss.ac.cn.

‡Corresponding author. Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and School of Mathematics Sciences, University of Chinese Academy of Sciences, Beijing 100149, China. E-mail address: jif@iss.ac.cn.
and $\mathcal{F}$-adapted essentially bounded processes; and $C^r_F([0, T]; L^r(\Omega; \mathcal{H}))$ denotes the Banach space consisting of all $\mathcal{F}$-adapted processes $X(\cdot)$ such that $X(\cdot) : [0, T] \to L^r_{\mathcal{F}_t}(\Omega; \mathcal{H})$ is continuous ($r \in [1, \infty]$). Similarly, one can define $C^m_F([0, T]; L^r(\Omega; \mathcal{H}))$ for any positive integer $m$. All of the above spaces are endowed with their canonical norms.

Beam is a kind of special structure widely existing in material mechanics and engineering mechanics, such as railway track, continuously supported piles, bridge support structure and slender wings of aircraft. Study of beam models may date back to the 18th century, when Bernoulli found that the curvature of an elastic beam at any point is proportional to the bending moment and Euler ([26]) studied the deformation of elastic beams under different load conditions. Since the Euler-Bernoulli beam theory has many applications in the practical engineering problems, research in this field have received lots of attention ([22, 24, 25]).

In order to simulate the large amplitude vibration of an elastic panel excited by aerodynamic forces, [3] introduced a stochastic Euler-Bernoulli beam equation, in which a force caused by random fluctuations was considered. The classical stochastic beam equation only involves random perturbations of external forces, which has the following form in one dimension:

$$
\begin{aligned}
\begin{cases}
    dy_t + y_{xxxx} dt = f dt + g dW(t) & \text{in } Q, \\
    y(0, t) = 0, \quad y_x(0, t) = 0 & \text{on } (0, T), \\
    y(1, t) = 0, \quad y_x(1, t) = 0 & \text{on } (0, T), \\
    y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x) & \text{in } (0, 1),
\end{cases}
\end{aligned}
$$

(1.1)

where $y$ denotes the lateral displacement of the beam, $(y_0, y_1)$ is the initial data, $f$ denotes the continuous excitation, and $g$ denotes the random perturbation.

In reality problems, due to air turbulence at high speed, both the pressure and the aerodynamic force are perturbed by random fluctuations. However, the random perturbation between velocity and displacement is ignored in the derivation of the classical stochastic beam equation (1.1). According to the dynamical theory of Brownian motions in [20] and the derivation process of the model in [16], we make a modification to classical stochastic beam equation. Stimulated by the uncertainty between velocity and displacement, and combing with the classical Euler-Bernoulli beam theory, we can get the following refined stochastic beam equation:

$$
\begin{aligned}
\begin{cases}
    dy_t = \dot{y} dt + \bar{f} dW(t) & \text{in } Q, \\
    d\dot{y} + y_{xxxx} dt = f dt + g dW(t) & \text{in } Q, \\
    y(0, t) = 0, \quad y_x(0, t) = 0 & \text{on } (0, T), \\
    y(1, t) = 0, \quad y_x(1, t) = 0 & \text{on } (0, T), \\
    y(x, 0) = y_0(x), \quad \dot{y}(x, 0) = \dot{y}_0(x) & \text{in } (0, 1),
\end{cases}
\end{aligned}
$$

(1.2)

where $y$ denotes the lateral displacement of the beam, $\dot{y}$ denotes the lateral velocity of the beam, $(y_0, \dot{y}_0)$ is the initial data, $\bar{f}$ denotes the uncertainty between velocity and displacement, $f$ denotes the continuous excitation, and $g$ denotes the random perturbations.

Carleman-type estimate was first introduced by Carleman in 1939 to study the uniqueness for elliptic equations in two dimension ([2]). It has become an important tool in studying
the uniqueness, control and inverse problems for deterministic partial differential equations ([1, 6, 10, 11, 27, 31] and the references therein). However, people know little about stochastic counterpart. We refer to [5, 14, 15, 23, 29, 30] for some known Carleman estimates of the stochastic partial differential equations.

This paper is devoted to establishing global Carleman estimates for refined stochastic beam equations, and applying these estimates to study two classes of important ill-posed problems: the exact controllability and inverse source problems for refined stochastic beam equations.

There are numerous works ([9, 12, 13, 18, 19, 28] and the references therein) addressing controllability and inverse source problems for deterministic beam equations. However, new difficulties arise in dealing with the related control and inverse problem of the stochastic counterpart, for example, the solution of a stochastic partial differential equation is non-differentiable with respect to noise variable, and the usual compactness embedding result is not valid.

The contribution and findings of this work are summarized as follows.

• We establish global Carleman estimates for the coupled stochastic equation in this paper. In order to study controllability of the two-order stochastic partial differential equation, Carleman estimates are obtained for stochastic parabolic equation in [23] and refined stochastic hyperbolic equation in [16]. In contrast, as a class of high order stochastic partial differential equation, stochastic beam equations have more higher-order terms to deal with than stochastic parabolic equations and stochastic hyperbolic equations when establishing Carleman estimates, and it is more complicated and difficult. The difference in controllability between the classical stochastic beam equations and the refined stochastic beam equations can be revealed by the estimates.

• Generally speaking, deterministic beam equation is exactly controllable by applying controls at the boundary or inside. But, we find the classical stochastic beam equation is not exactly controllable at any time, even if controls are acted everywhere on both the drift term, the diffusion term and boundary. These controls are the most powerful control actions that one can introduce into equation. However, once the model is reasonably modified, we can achieve exact controllability with a minimum number of controls.

• As the other application of Carleman estimates, we also establish the uniqueness of an inverse source problem for refined stochastic beam equations. In [29], Carleman estimates of classical stochastic beam equations are established to study the inverse source problems. Compared with the existing result, we choose weight functions with singularity in the Carleman estimate. This estimate requires less observation information, and does not require any information about initial and terminal values. Furthermore, multiple source terms, including drift source terms and diffusion source terms, can be determined simultaneously.

The rest of this paper is organized as follows. Section 2 presents the main results in this paper. In Section 3, we give a pointwise weighted identify for refined stochastic beam operator. Section 4 is devoted to proving the exact controllability results for refined stochastic beam equation, while Section 5 is on the detailed proof of the inverse problem result. Finally, we summarise the paper and discuss future topic that are worth investigating in Section 6.

For notational simplicity, we give some notations:

\[ \mathcal{H}_1 = (H^4(0, 1) \cap H^2_0(0, 1)) \times H^2_0(0, 1), \quad \mathcal{H}_2 = L^2(0, 1) \times H^{-2}(0, 1), \]
Consider the following refined stochastic beam equation:

\[ H_3 = C_F([0, T]; L^2(\Omega; L^2(0, 1))) \cap C_F([0, T]; L^2(\Omega; H^{-2}(0, 1))), \]

\[ H_4 = C_F([0, T]; L^2(\Omega; L^2(0, 1))) \times C_F([0, T]; L^2(\Omega; H^{-2}(0, 1))), \]

\[ H_5 = L^2_0(\Omega; C([0, T]; H^4(0, 1) \cap H^2_0(0, 1))) \times L^2_0(\Omega; C([0, T]; H^2_0(0, 1))), \]

\[ H_6 = C_F([0, T]; L^2(\Omega; H^{-2}(0, 1))) \times C_F([0, T]; L^2(\Omega; H^{-2}(0, 1))). \]

## 2 Main results

Consider the following refined stochastic beam equation:

\[
\begin{aligned}
\frac{d\hat{y}}{dt} &= (\hat{y} + f_1)dt + g_1 dW(t) & \text{in } Q, \\
\frac{d\hat{y}}{dt} + y_{xxx} dt &= f_2 dt + g_2 dW(t) & \text{in } Q, \\
y(0, t) &= y_x(0, t) = 0 & \text{on } (0, T), \\
y(1, t) &= y_x(1, t) = 0 & \text{on } (0, T), \\
y(x, 0) &= y_0(x), \quad \hat{y}(x, 0) = \hat{y}_0(x) & \text{in } (0, 1),
\end{aligned}
\]  

(2.1)

where \((y_0, \hat{y}_0) \in H_1, f_1, f_2, g_2 \in L^2_0(0, T; H^2_0(0, 1), g_1 \in L^2(0, T; H^4(0, 1) \cap H^2_0(0, 1)). \)

Similar to the discussion in [4, 17], for any \((y_0, \hat{y}_0) \in H_1, f_1, f_2, g_2 \in L^2_0(0, T; H^2_0(0, 1)), \)
and \(g_1 \in L^2_0(0, T; H^4(0, 1) \cap H^2_0(0, 1)), (2.1) \) admits a unique solution \((y, \hat{y}) \in H_5. \)

Here, we introduce some auxiliary functions. Let \(\psi_1, \psi_2 \in C^4([0, 1]) \) satisfy that

\[ \psi_1(x) > 0 \text{ in } [0, 1], \quad \psi_{1,x}(x) < 0 \text{ in } [0, 1], \]

and

\[ \psi_2(x) > 0 \text{ in } (0, 1), \quad \psi_2(0) = \psi_2(1) = 0, \quad |\psi_{2,x}| > 0 \text{ in } (0, 1) \setminus G_1. \]

For any parameters \(\lambda, \mu \geq 1\) and \(i = 1, 2,\) put

\[ \theta_i = e^\ell_i, \quad \ell_i = \lambda \eta_i, \quad \eta_i = \frac{e^{\mu \psi_i} - e^{2\mu |\psi_i| C([0, 1])}}{t^2(T-t)^2}, \quad \varphi_i = \frac{e^{\mu \psi_i}}{t^2(T-t)^2}. \]

By establishing a suitable a weighted identify and choosing a weight function \(\theta_1 = \theta_1(x, t), \)
we have the following global Carleman estimate for (2.1).

**Theorem 2.1** There are two positive constants \(\mu_1, \lambda_1(\mu), \)
such that for all \(\mu \geq \mu_1, \lambda \geq \lambda_1(\mu), \)
and any solution \((y, \hat{y}) \in H_5 \) to (2.1), it holds that

\[
\mathbb{E} \int_Q \left[ \lambda^7 \mu^8 \varphi_1^2 y^2 + \lambda^5 \mu^6 \varphi_1 f_1^2 + \lambda^3 \mu^4 \varphi_3(y^2_x + \hat{y}^2) + \lambda^2 \mu^2 \varphi_1(y^2_{xxx} + \hat{y}^2_x) \right] dx dt \\
\leq C \mathbb{E} \int_Q \left[ \lambda^7 \mu^6 \varphi_1^2 f_1^2 + \lambda^4 \mu^4 \varphi_1^2 f_1^2 + \lambda^2 \mu^2 \varphi_1 f_1^2 + f_2^2 \\
+ \lambda^6 \varphi_1^2 g_1^2 + \lambda^4 \varphi_1^2 g_1^2 + \lambda^2 \varphi_1 g_1^2 + g_1^2 + \lambda^2 \mu^2 \varphi_1 g_1^2 \right] dx dt \\
+ C \mathbb{E} \int_0^T \left[ \lambda \mu \varphi_1 y^2_{xxx}(0, t) + \lambda^3 \mu^3 \varphi_3^2 y^2_{xx}(0, t) \right] dt.
\]
Moreover, the other global Carleman estimate for (2.1) is established by taking different weight function \( \theta_2 = \theta_2(x, t) \).

**Theorem 2.2** There are two positive constants \( \mu_2 \) and \( \lambda_2(\mu) \), such that for all \( \mu \geq \mu_2 \), \( \lambda \geq \lambda_2(\mu) \), and any solution \((y, \dot{y}) \in \mathcal{H}_2 \) to (2.1), it holds that

\[
\mathbb{E} \int_Q \theta_2^2 \left[ \lambda^2 \mu^2 \phi_2^2 y^2 + \lambda^5 \mu^6 \phi_2^3 y^2 + \lambda^3 \mu^4 \phi_2^3 (y_x^2 + \dot{y}^2) + \lambda \mu^2 \phi_2 (y_{xxx}^2 + \dot{y}_x^2) \right] dx dt \\
\leq C \mathbb{E} \int_Q \theta_2^2 \left[ \lambda^6 \mu^6 \phi_2^5 f_1^2 + \lambda^4 \mu^4 \phi_2^4 f_1^2 + \lambda^2 \mu^2 \phi_2^2 f_1^2 + f_2^2 \right] dx dt \\
+ \lambda^4 \phi_2 g_1^2 + \lambda^4 \phi_2 g_1^2 + \lambda^2 \phi_2 g_1^2 + g_1^2 + \lambda^2 \mu^2 \phi_2 g_2^2 dx dt \\
+ C \mathbb{E} \int_0^T \theta_2^2 \left[ \lambda^7 \mu^8 \phi_2^2 y^2 + \lambda^6 \mu^6 \phi_2^2 y^2 + \lambda^5 \mu^4 \phi_2^4 y^2 + \lambda^2 \mu^2 \phi_2 y_{xxx}^2 \right] dx dt.
\]

(2.3)

**Remark 2.1** By choosing different weighted functions, two Carleman estimates (2.2) and (2.3) are established. The main difference between the above estimates is the local information on the right-hand of the inequalities, one is internal information and the other is boundary information.

**Remark 2.2** Compared with the results in [29], there are no terms with respect to initial and terminal data on the right-hand of (2.2) and (2.3). It means that the additional conditions on the initial and terminal data are not required any more by the above Carleman estimates, and therefore, the known observations may be reduced. The main reason is the choice of the “singular” weight function \( \theta_1 \) with respect to time.

The first application of the above Carleman estimates is the exact controllability of the refined stochastic beam equation. Controllability means to find a control such that the corresponding state of the considered system achieves a prescribed goal at a given time. Consider the following controlled stochastic beam equation:

\[
\begin{align*}
\frac{dy}{dt} &= \dot{y} dt + f dW(t) & \text{in } Q, \\
\frac{d\dot{y}}{dt} + y_{xxx} dt &= (a_1 y + a_2 y_x + a_3 g) dt + g dW(t) & \text{in } Q, \\
y(0, t) &= h_1, \quad y_x(0, t) = h_2 & \text{on } (0, T), \\
y(1, t) &= h_3, \quad y_x(1, t) = h_4 & \text{on } (0, T), \\
y(x, 0) &= y_0(x), \quad \dot{y}(x, 0) = \dot{y}_0(x) & \text{in } (0, 1),
\end{align*}
\]

where \((y_0, \dot{y}_0)\) is the initial data, \(a_1, a_2, a_3 \in L^\infty(0, T; L^\infty(0, 1))\), \(a_2 \in L^\infty(0, T; W^{1, \infty}(0, 1))\), and \(f \in L^2_T(0, T; L^2(0, 1))\). \(g \in L^2_T(0, T; H^{-2}(0, 1))\), \(h_i \in L^2_T(0, T) \quad (i = 1, 2, 3, 4)\) are controls.

Similar to the method used in [16], for any \((y_0, \dot{y}_0) \in \mathcal{H}_2\), \(f \in L^2_T(0, T; L^2(0, 1))\) and \(g \in L^2_T(0, T; H^{-2}(0, 1))\). (2.4) admits a unique transposition solution \((y, \dot{y}) \in \mathcal{H}_4\) (the definition of transposition solution can be found in [16]).

**Remark 2.3** The term \(a_3 g\) in (2.4) reflects the effect of the control \(g\) in the diffusion term to the drift term in some way. It is the side effect of the control \(g\). In other words, if a control \(g\) is put in the diffusion term, then \(a_3 g\) appears passively in the drift term.
Now, we give the definition of the exact controllability for (2.4).

Definition 2.1 The system (2.4) is called exactly controllable at time $T$ if for any $(y_0, \hat{y}_0) \in \mathcal{H}_2$ and $(y_1, \hat{y}_1) \in L^2(0, T; \mathbb{R}) \times L^2(0, T; \mathbb{H}^{-2}(0, 1))$, one can find a pair of control

$$(f, g, h_1, h_2) \in L^2(0, T; L^2(0, 1)) \times L^2(0, T; H^{-2}(0, 1)) \times (L^2(0, T))^2,$$

such that the solution $(y, \hat{y})$ of (2.4) corresponding to the above controls and $h_3 = h_4 = 0$ satisfies that $(y(\cdot, T), \hat{y}(\cdot, T)) = (y_1, \hat{y}_1)$.

There are some controllability results for stochastic systems ([5, 7, 8, 16, 23]). Similar to the results in [16], we find that the exact controllability of classical stochastic beam system is fails for any $T > 0$, even if the controls are acted everywhere on the whole domain $(0, 1)$ and the boundary. This is quite different from the well-known controllability property of deterministic beam system. But, after reasonable modifications of the stochastic beam system, the following exact controllability result is obtained:

Theorem 2.3 The system (2.4) is exactly controllable at any time $T > 0$.

Remark 2.4 The boundary controls $h_1$ and $h_2$ in (2.4) are imposed on the boundary point 0. The above result still holds if they are effective on the boundary point 1 for (2.4).

Remark 2.5 By means of the Carleman estimate (2.3), the exact controllability of (2.4) also can be established if the controls $f, g$ are acted in the diffusion terms, and the local internal control $h_5 \in L^2(0, T; H^{-3}(G_0))$ is put in the drift term for the second equation of (2.4).

Remark 2.6 It is worth noting that four controls $f$ and $g$ (in the diffusion terms), $h_1$ and $h_2$ (on the boundary) are required in (2.4) to derive the exact controllability. Moreover, the controls $f$ and $g$ in diffusion terms are active on the whole domain. In fact, these conditions cannot be weakened. More details will be given in Theorem 4.3.

Remark 2.7 Compared with [16], the background of our problem is different. In [16], the exact controllability of the stochastic wave equation is investigated. In this paper, different weight functions with singularity are chosen, and the Carleman estimates containing local interior and boundary information are established. By the Carleman estimates, not only the exact controllability of the refined stochastic beam equations, but also an inverse source problem are studied.

The other application of the Carleman estimates in this paper is the inverse source problem for the following refined stochastic beam equation:

$$\begin{cases}
  dy = [\hat{y} + A(t)R(x, t)]dt + [c_1y + B(t)R(x, t)]dW(t) & \text{in } Q, \\
  \hat{dy} + y_{xxxx}dt = [c_2y + c_3y_x + H(t)R(x, t)]dt + [c_4y + P(t)R(x, t)]dW(t) & \text{in } Q, \\
  y(0, t) = y_x(0, t) = 0 & \text{on } (0, T), (2.5) \\
  y(1, t) = y_x(1, t) = 0 & \text{on } (0, T), \\
  y(x, 0) = y_0(x), \quad \hat{y}(x, 0) = \hat{y}_0(x) & \text{in } (0, 1),
\end{cases}$$
where the initial data \((y_0, \hat{y}_0) \in H_1\), the coefficients \(c_1 \in L^\infty_F(0,T;W^{4,\infty}(0,1))\), \(c_2, c_4 \in L^\infty_F(0,T;W^{2,\infty}(0,1))\), \(c_3 \in L^\infty_F(0,T;W^{3,\infty}(0,1))\), and \(A, B, H, P \in L^2_F(0,T)\) are four unknown sources.

For suitably given function \(R\), our inverse problem is to determine four source functions \(A(\cdot), B(\cdot), H(\cdot)\) and \(P(\cdot)\) by means of the known observation information of the boundary and the interior domain. The main result on the uniqueness of (2.5) is as follows:

**Theorem 2.4** Assume that \(R \in C^5(\overline{Q})\) and \(R \neq 0\) in \(\overline{Q}\). Let \((y, \hat{y}) \in H_5\) be any solution to (2.5). If
\[
\begin{align*}
(y_{xx}(0,t) &= y_{xx}(1,t) = 0 \quad \text{on } (0,T), \\
y &= 0 \quad \text{in } G_0 \times (0,T),
\end{align*}
\tag{2.6}
\]
then
\[A(t) = B(t) = H(t) = P(t) = 0 \quad \text{for all } t \in [0,T], \ P\text{-a.s.}\]

**Remark 2.8** Compared with [29], by choosing the singular weight functions, we establish the different Carleman estimates. From the Carleman estimate, we can determine four sources simultaneously, not only the drift source terms, but also the diffusion terms. And, there is no restriction \(\sqrt{T-t}\) in the right-hand side of diffusion source term. Furthermore, the requirements for initial and final values are removed from the observation information.

**Remark 2.9** In (2.5), all sources have the form of separated variables. If the sources in drift term are general, it is easy to show there are counterexamples ([15]). It is interesting to consider the general sources in the diffusion term. More precisely, in (2.5), \(B(t)R(x,t)\) and \(P(t)R(x,t)\) are replaced by \(B(x,t)\) and \(P(x,t)\), respectively. But, this remains to be done and it is our future work.

**Remark 2.10** It seems a bit strange that the measurements in local domain \(G_0\) and on the boundary simultaneously are needed in terms of mathematical theory, but it is acceptable in practical application. In the real wave models, internal information is not easy to obtain. However, it seems not difficult for the real beam models. For example, we can put a tiny sensor in the wing of an airplane, or in a gap of the bridge structure, to measure the information inside a beam model.

**Remark 2.11** In [29], the known observations are in the local area. If the known information is only on the local boundary, it is still open and we will consider it in a forthcoming paper.

### 3 A pointwise weighted identify

This section is devoted to establishing a weighted identify for stochastic beam-like operator \(d\hat{y} + y_{xxxx} dt\), which will play a key role in what follows.
Assume that $\Psi \in C^3(\mathbb{R}^n \times (0, T))$, $\ell \in C^5(\mathbb{R}^n \times (0, T))$, and put

$$
\begin{align*}
\Psi &= -9\ell_x^2\ell_{xx}, \\
\Phi &= \ell_x^4 - 6\ell_x^2\ell_{xx} + 3\ell_x^2 + 4\ell_x\ell_{xxx} - \ell_{xxxx} + \ell_t^2 - \ell_{tt}, \\
A &= 3(\Psi\ell_x^2 - \Psi\ell_{xx})_{xx} + (\ell_t\Phi - \ell_t\Psi)_t + 2(\ell_x\Phi - \ell_x\Psi)_{xxx} + \Psi(\Phi - \Psi) \\
&+ 2[D(\Phi - \Psi)]_x, \\
B &= 12[D(\ell_x^2 - \ell_{xx})]_x - 6(\Psi\ell_x^2 - \Psi\ell_{xx}) - 6(\ell_t\ell_x^2 - \ell_t\ell_{xx})_t - 6(\ell_x\Phi - \ell_x\Psi)_x, \\
D &= \ell_x^3 + 3\ell_x\ell_{xx} + \ell_{xxx}.
\end{align*}
$$

Lemma 3.1 Let $y$ be an $H^4(\mathbb{R})$-valued semimartingale, $\hat{y}$ be an $H^2(\mathbb{R})$-valued semimartingale, and

$$
dy = \hat{y}dt + f_1dt + g_1dW(t) \text{ in } Q, \tag{3.1}
$$

for some $f_1 \in L^2_F(0, T; H^2(0, 1))$, $g_1 \in L^2_F(0, T; H^4(0, 1) \cap H^2(0, 1))$. Set $z = \theta y$, $\hat{z} = \theta\hat{y} + \ell_t z$, and $\theta = e^\ell$. Then, for a.e. $x \in (0, 1)$, and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$
\theta I(dy + y_{xxxx} dt) = dM + V_xdt + I^2dt + 4(\ell_x\hat{z}dz_{xx})_x + 6\ell_{xx}\hat{z}_x^2 dt + 2\ell_{xx}\hat{z}_xx dt + A\hat{z} dt + B\hat{z}_x dt \\
+ (\ell_{tt} - 2\ell_{xxxx} - 2D_x - \Psi)\hat{z}^2 dt + [\Psi - 6D_x + \ell_{tt} + 12(\ell_x^2 - \ell_x\ell_{xx})]_x\hat{z}_x dt + U dt + P dt + \ell_t(d\hat{z})^2 + \ell_t(\Phi - \Psi)(dz)^2 - 6\ell_t(\ell_x^2 - \ell_x\ell_{xx})(dz_x)^2 + \ell_t(dz_{xx})^2 \\
- \Psi dz\hat{z} + 4\ell_x dz\hat{z}dz_{xx} + 4Dd\hat{z}dz_x + \left\{2\ell_t(\Phi - \Psi)z + \Psi\hat{z} - I\ell_t\theta g_1 + 4D\hat{z}\theta g_1_x - 12(\ell_t\ell_x^2 - \ell_t\ell_{xx})z_x(\theta g_1)_x + [2\ell_tz_{xx} - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)](\theta g_1)_x\right\} dW(t),
$$

where

$$
I = -2\ell_t\hat{z} - 4\ell_x\hat{z}_{xxx} - 4D_x + \Psi z,
$$

$$
M = -\ell_t\hat{z}_x^2 - (4\ell_xz_{xxx} + 4Dz_x - \Psi z)\hat{z} - \ell_t\hat{z}_x^2 + 6\ell_t(\ell_x^2 - \ell_x\ell_{xx})\hat{z}_x^2 - \ell_t(\Phi - \Psi)\hat{z}_x^2, \\
V = -4\ell_{xx}\hat{z}_{xx} - 2\ell_x\hat{z}_x^2 + 2(D + \ell_{xxx})\hat{z}_x^2 - 2\ell_t\hat{z}_{xxx} - 2\ell_x\hat{z}_xx - 4Dz_{xxx} + \Psi z_{xxx} + 2[D(\Phi - \Psi)z_{xx} + 2\ell_t\hat{z}_{xx}z_{xx} + 2\ell_t\hat{z}_{xxx}z_{xx} - 12(\ell_x^2 - \ell_x\ell_{xx})\hat{z}_x] \\
- 4(\ell_x\Phi - \ell_x\Psi)z_{xx} + 2[\ell_x\Phi - \ell_x\Psi - 6D(\ell_x^2 - \ell_x\ell_{xx})]_x\hat{z}_x + 8(\Psi^2 - \Psi\ell_{xx})(z_{xx})_x \\
+ 4(\ell_x\Phi - \ell_x\Psi)z_{xx} - 2D(\Phi - \Psi) + 3(\Psi^2 - \Psi\ell_{xx})_x + 2(\ell_x\Phi - \ell_x\Psi)(\ell_x\Psi)_{xx} - 2\ell_tz_xz_{xx} \\
+ 6\ell_t\hat{z}_x z_{xxx} - \Psi z_{xxx},
$$

$$
P = (\Psi_x - 4D_{xx})z_{xx} + [4\ell_t + 12(\ell_t\ell_x^2 - \ell_t\ell_{xx})]_x\hat{z}_{xx} - \Psi_t\hat{z}z - 2\ell_tz_xz_{xx} \\
+ 6\ell_t\hat{z}z_{xxx} - \Psi z_{xxx},
$$

$$
U = [-\ell_t + 2(\Phi - \Psi)\ell_t z - \Psi \hat{z}]\theta f_1 + [4D\hat{z} - 12(\ell_t\ell_x^2 - \ell_t\ell_{xx})z_x](\theta f_1)_x \\
+ [2\ell_tz_{xx} - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)](\theta f_1)_x,
$$

and $(dz)^2$, $(d\hat{z})^2$ denote the quadratic variation processes of $z$ and $\hat{z}$, respectively.
Proof. See Appendix A for the proof.

Then, from the weighted identify (3.2), and by properly choosing different weight functions, one can establish the global Carleman estimates (2.2) and (2.3). The detailed proofs of Theorem 2.1 and 2.2 are provided in Appendixes B and C, respectively.

4 Controllability

In this section, we study the controllability problem for the classical and the refined stochastic beam systems. First, consider the following classical stochastic beam system:

\[
\begin{aligned}
&\text{d}y_t + y_{xxxx}\text{d}t = (a_1y + a_2y_x + f)\text{d}t + g\text{d}W(t) \quad \text{in } Q, \\
y(0, t) = h_1, & \quad y_x(0, t) = h_2 \quad \text{on } (0, T), \\
y(1, t) = h_3, & \quad y_x(1, t) = h_4 \quad \text{on } (0, T), \\
y(x, 0) = y_0(x), & \quad y_t(x, 0) = y_1(x) \quad \text{in } (0, 1),
\end{aligned}
\]

(4.1)

where the initial datum \((y_0, y_1) \in H_2\), \(y\) is the state variable, \(a_1 \in L^\infty(0, T; L^\infty(0, 1))\), \(a_2 \in L^\infty(0, T; W^{1,\infty}(0, 1))\), and \(f, g \in L^2_F(0, T; H^{-2}(0, 1))\), \(h_i \in L^2_F(0, T) (i = 1, 2, 3, 4)\) are controls.

Similar to the method used in [16], for any \((y_0, y_1) \in H_2\), \(f, g \in L^2_F(0, T; H^{-2}(0, 1))\), and \(h_i \in L^2_F(0, T) (i = 1, 2, 3, 4)\), (4.1) admits a unique solution \(y \in H_3\).

Recalling the definition of the exact controllability for (4.1).

**Definition 4.1** The system (4.1) is called exactly controllable at time \(T\) if for any \((y_0, y_1) \in H_2\), and \((y'_0, y'_1) \in L^2_F(\Omega; L^2(0, 1)) \times L^2_F(\Omega; H^{-2}(0, 1))\), one can find a pair of control

\[(f, g, h_1, h_2, h_3, h_4) \in L^2_F(0, T; H^{-2}(0, 1)) \times L^2_F(0, T; H^{-2}(0, 1)) \times (L^2_F(0, T))^4,\]

such that the solution \(y\) to (4.1) satisfies that \((y(\cdot, T), y_t(\cdot, T)) = (y'_0, y'_1)\).

Let us give the negative controllability result for (4.1).

**Theorem 4.1** The system (4.1) is not exactly controllable for any \(T > 0\).

The proof of Theorem 4.1 is similar to Theorem 2.1 in [16], which we omit here. We show that the classical stochastic system (4.1) is exactly controllable, even if the controls acted everywhere on the domain \((0, 1)\) and boundary. Obviously, it is quite different from the well-known controllability results of deterministic beam systems.

Next, as an application of the Carleman estimate (2.2), the controllability problems of the refined system (2.4) is considered. According to duality argument, the exact controllability problem of a stochastic partial differential equation is usually transformed into the observability estimate of the corresponding adjoint equation. But, generally speaking, it is easier to establish an observability estimate for a forward equation than to prove an observability estimate for a backward one in the stochastic counterparts. Then, we transform the exact
controllability problem of (2.4) into the exact controllability problem of a backward stochastic beam system. Therefore, the following controlled backward stochastic beam system is introduced:

\[
\begin{aligned}
dp &= \dot{p}dt + PdW(t) \\
d\hat{p} + p_{xxx}dt &= (a_1 p + a_2 p_x + a_3 \hat{P})dt + \hat{P}dW(t) & \text{in } Q, \\
p(0, t) &= h_1(t), \quad p_x(0, t) = h_2(t) & \text{on } (0, T), \\
p(1, t) &= 0, \quad p_x(1, t) = 0 & \text{on } (0, T), \\
p(x, T) = p_T(x), \quad \hat{p}(x, T) = \hat{p}_T(x) & \text{in } (0, 1),
\end{aligned}
\]

where \((p_T, \hat{p}_T) \in L^2_F(\Omega; L^2(0, 1)) \times L^2_F(\Omega; H^{-2}(0, 1)), (p, \hat{p}, P, \hat{P})\) are the states, \((h_1, h_2)\) are controls, \(a_1, a_3 \in L^2_F(0, T; L^\infty(0, 1)), \) and \(a_2 \in L^\infty_F(0, T; W^{1, \infty}(0, 1))\).

By [17], for any \((p_T, \hat{p}_T) \in L^2_F(\Omega; L^2(0, 1)) \times L^2_F(\Omega; H^{-2}(0, 1))\) and \((h_1, h_2) \in (L^2_F(0, T))^2\), (4.2) admits a unique transposition solution \((p, \hat{p}, P, \hat{P}) \in H_b\).

Now, we give the definition of the exact controllability for (4.2).

**Definition 4.2** The system (4.2) is called exactly controllable at time \(T\) if for any \((p_T, \hat{p}_T) \in L^2_F(\Omega; L^2(0, 1)) \times L^2_F(\Omega; H^{-2}(0, 1))\) and \((p_0, \hat{p}_0) \in H_2\), one can find a pair of control \((h_1, h_2) \in (L^2_F(0, T))^2\), such that the solution \((p, \hat{p}, P, \hat{P})\) to (4.2) satisfies that \((p(\cdot, 0), \hat{p}(\cdot, 0)) = (p_0, \hat{p}_0)\).

By the Definitions 2.1 and 4.2, we have the following result concerning the relationship between solutions of (2.4) and (4.2).

**Proposition 4.1** One the one hand, if \((y, \dot{y})\) is a solution of (2.4), then \((p, \hat{p}, P, \hat{P}) = (y, \dot{y}, f, g)\) is a solution of (4.2) with the final data \((p_T, \hat{p}_T) = (y(T), \dot{y}(T))\). On the other hand, if \((p, \hat{p}, P, \hat{P})\) is a solution of (4.2), then \((y, \dot{y}) = (p, \hat{p})\) is a solution of (2.4) with the initial data \((y_0, \dot{y}_0) = (p(0), \hat{p}(0))\), and nonhomogeneous terms \((f, g) = (P, \hat{P})\).

From the Proposition 4.1, the following result holds:

**Proposition 4.2** The system (2.4) is exactly controllable at time \(T\) if and only if the system (4.2) is exactly controllable at time \(T\).

It is show that the exact controllability problem of the forward stochastic beam system (2.4) can be transformed into the exact controllability problem of the backward system (4.2). By duality theory, in order to study the exact controllability of (4.2), introduce the following stochastic beam system:

\[
\begin{aligned}
dz &= \dot{z}dt - a_3 z dW(t) & \text{in } Q, \\
d\hat{z} + z_{xxx}dt &= [(a_1 - a_{2,x})z - a_{2zz}z_x]dt & \text{in } Q, \\
z(0, t) = z_x(0, t) &= 0 & \text{on } (0, T), \\
z(1, t) = z_x(1, t) &= 0 & \text{on } (0, T), \\
z(x, 0) = z_0(x), \quad \dot{z}(x, 0) = \dot{z}_0(x) &= \dot{z}_0(x) & \text{in } (0, 1),
\end{aligned}
\]
where \((z_0, \hat{z}_0) \in \mathcal{H}_1\).

The following result shows that the exact controllability of (4.2) is equivalent to an observability estimate of (4.3).

**Proposition 4.3** The system (4.2) is exactly controllable at time \(T\) if and only if there is a constant \(C > 0\) such that for all \((z, \hat{z}) \in \mathcal{H}_5\), it holds that

\[
|z_0, \hat{z}_0|_{\mathcal{H}_1}^2 \leq C \mathbb{E} \int_0^T \left[ z_{xx}^2(0, t) + z_{xxx}^2(0, t) \right] dt, \quad \forall (z_0, \hat{z}_0) \in \mathcal{H}_1. \tag{4.4}
\]

The proof of Proposition 4.3 is based on duality argument and can be found in [16]. From Proposition 4.3, the null controllability result can be reduced to an observability estimate. Now, we give an observability estimate for (4.3).

**Theorem 4.2** There is a constant \(C > 0\), such that all solutions \((z, \hat{z}) \in \mathcal{H}_5\) to (4.3) satisfy (4.4).

**Proof.** By (4.3) and Theorem 2.1, there exist \(\mu_1 > 0\) and \(\lambda_1 = \lambda_1(\mu)\), such that for all \(\mu \geq \mu_1\) and \(\lambda \geq \lambda_1(\mu)\),

\[
\mathbb{E} \int_0^T \left[ \lambda^2 \mu_1^2 \varphi_1^2 \varphi_1 z^2 + \lambda^1 \mu \varphi_1^2 \varphi_1 z^2 + \lambda^2 \mu_1^2 \varphi_1^2 (z_{xx}^2 + \hat{z}^2) + \lambda^3 \mu_1^2 \varphi_1 (z_{xxx} + \hat{z}_x^2) \right] dx dt \leq C \mathbb{E} \int_0^T \lambda \mu \varphi_1 z_{xxx}^2(0, t) + \lambda^3 \mu_1^2 \varphi_1 z_{xxx}^2(0, t) dt. \tag{4.5}
\]

Set

\[
\mathcal{E}(t) = \frac{1}{2} \mathbb{E} \int_0^1 \left[ z^2(x, t) + \hat{z}^2(x, t) + z_x^2(x, t) + z_{xx}^2(x, t) \right] dx, \quad t \in [0, T].
\]

Then, by Itô’s formula, for any \(s, t\) satisfy \(0 \leq s \leq t \leq T\), we have

\[
\mathcal{E}(t) - \mathcal{E}(s) = \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left[ 2zdz + (dz)^2 \right] dx + \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left[ 2\hat{z}d\hat{z} + (d\hat{z})^2 \right] dx + \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left[ 2z_x dz_x + (dz_x)^2 \right] dx + \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left[ 2z_{xx} dz_{xx} + (dz_{xx})^2 \right] dx
\]

\[
= \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left\{ 2\hat{z} + (a_3 z)^2 + a_1 a_2 z^2 + (a_3 z_x)^2 + 2z_{xx} \hat{z}_{xx} + 2(a_3 z_{xx})^2 \right\} dx dt
\]

\[
= \frac{1}{2} \mathbb{E} \int_s^t \int_0^1 \left\{ 2\hat{z} + (a_3 z)^2 - 2z_{xx} \hat{z} + (a_3 z_x)^2 + 2(a_3 z_{xx})^2 \right\} dx dt,
\]

11
and hence,
\[ \mathcal{E}(s) \leq \mathcal{E}(t) + CE \int_s^t \int_0^1 (z^2 + \dot{z}^2 + z_x^2 + z_{xx}^2)dx\,d\tau. \]

By Gronwall’s inequality, one can obtain that
\[ \mathcal{E}(s) \leq C\mathcal{E}(t). \] (4.6)

Taking \( \lambda = \lambda_1, \mu = \mu_1 \) in (4.5), and recalling the definitions of \( \theta_1, \varphi_1 \), we have
\[ E \int_{T/4}^{3T/4} \int_0^1 \left[ z^2 + z_x^2 + (z_{xx}^2 + \dot{z}^2) \right] dx\,dt \leq C E \int_0^T \left[ z_{xxx}^2(0,t) + z_{xx}^2(0,t) \right] dt. \] (4.7)

By (4.6) and (4.7), we complete the proof of Theorem 4.2. \( \Box \)

**Proof of Theorem 2.3.** It follows from Proposition 4.3 and Theorem 4.2 immediately. \( \Box \)

Although it is necessary to put controls \( f \) and \( g \) on the whole domain, one may suspect that Theorem 4.1 is trivial. However, we have the following negative result:

**Theorem 4.3** The system (2.4) is not exactly controllable at any time \( T > 0 \) provided that one of the following three conditions is satisfied:
1) \( \text{supp} f \subseteq G_2, \text{where } G_2 \subseteq (0,1); \)
2) \( \text{supp} g \subseteq G_2, \text{where } G_2 \subseteq (0,1); \)
3) \( h_1 = h_2 = 0. \)

**Remark 4.1** Although boundary controls \( h_1, h_2 \) cannot be dropped simultaneously in (2.4), but it is worth studying if one of them can be removed. It seems to be true from the existing controllability results of deterministic beam systems.

From the above negative result, we find that none of the two internal controls \( f, g \) and boundary controls can be ignored, and internal controls must be effective everywhere in the domain \((0,1)\). The proof of Theorem 4.3 is based on the contradiction argument, and the known conclusions ([21], Lemma 2.1). We omit it here, since it is similar to the proof of Theorem 2.3 in [16].

In this paper, we prove that (2.4) is exactly controllable, but (4.1) is not exactly controllable at any time. Therefore, from the viewpoint of control theory, the refined system (2.4) is a more reasonable model than the classical system (4.1).

### 5 Inverse problem

In this section, we are devoted to proving the inverse source result of the refined stochastic beam equation through the Carleman estimate (2.3).
Proof of Theorem 2.4. Let \((y, \dot{y})\) be the solution of (2.5). Set \(y = Rh, \dot{y} = \dot{R}h\). Then, it can be seen that \((h, \dot{h})\) satisfies that

\[
\begin{aligned}
\dot{h} &= h + A(t) - \frac{R_t}{R} \dot{h} + (c_1 h + B(t))dW(t) \quad \text{in } Q, \\
\frac{d}{dt} h + h_\infty dt &= \left[ c_2 h + c_3 h_x + c_3 \frac{R_x}{R} h - \frac{R_t}{R} \dot{h} - 4R_x h_\infty - 6R_{xx} h_{xx} \right. \\
&- \left. 4R_{xxx} h_x - \frac{R_{xxxx}}{R} h + H(t) \right] dt + \left[ c_4 h + P(t) \right] dW(t) \quad \text{in } Q, \\
h(0, t) &= h(0, t) = 0 \\
h(1, t) &= h(1, t) = 0 \\
h(x, 0) &= h_0(x) = \frac{y_0(x)}{R(x, 0)}, \quad \dot{h}(x, 0) = \dot{h}_0(x) = \frac{\dot{y}_0(x)}{R(x, 0)} \quad \text{in } (0, 1).
\end{aligned}
\]

Put \(u = h_x, \dot{u} = \dot{h}_x\). Then, by (2.6) and (5.1), \((u, \dot{u})\) solves

\[
\begin{aligned}
u &= \frac{R}{R} u - (c_2 \frac{R}{R} h) dt + (c_1 u + c_1 h) dW(t) \quad \text{in } Q, \\
\frac{d}{dt} u + u_\infty dt &= \left[ \tilde{E}_1(u, \dot{u}) + \tilde{F}_1(h, \dot{h}) \right] dt \\
&+ (c_4 u + c_4 h) dW(t) \quad \text{in } Q, \\
u(0, t) &= u_x(0, t) = 0 \\
u(1, t) &= u_x(1, t) = 0 \\
u(x, 0) &= h_{0,x}(x), \quad \dot{u}(x, 0) = \dot{h}_{0,x}(x) \quad \text{in } (0, 1),
\end{aligned}
\]

where

\[
\tilde{E}_1(u, \dot{u}) = c_2 u + c_3 u_x + c_3 \frac{R_x}{R} u - \frac{R_t}{R} \dot{u} - 4R_x u_\infty - 6R_{xx} u_{xx} - 4R_{xxx} u_x
\]

\[
- \frac{R_{xxxx}}{R} u + c_3 u_x - \left( 4R_x \frac{R}{R} \right) u_{xx} - \left( 6R_{xx} \frac{R}{R} \right) u_x - \left( 4R_{xxx} \frac{R}{R} \right) u,
\]

and

\[
\tilde{F}_1(h, \dot{h}) = c_2 h + \left( c_3 \frac{R_x}{R} \right) h - \left( \frac{R_t}{R} \right) \dot{h} - \left( \frac{R_{xxxx}}{R} \right) h.
\]

By means of \(y(0, t) = \dot{y}(0, t) = 0\) in \((0, T)\), we have

\[
h = \int_0^x h_x(\eta, t) d\eta = \int_0^x u(\eta, t) d\eta,
\]

\[
\dot{h} = \int_0^x \dot{h}_x(\eta, t) d\eta = \int_0^x \dot{u}(\eta, t) d\eta.
\]
Therefore, together with (5.2), \((u, \dot{u})\) satisfies
\[
\begin{cases}
    \text{d}u = \left[ \dot{u} - \frac{R_t}{R} u - \frac{R_t}{R} \int_0^x u(\eta, t) \text{d}\eta \right] \text{d}t \\
    \quad + \left[ c_1 u + c_1 x \int_0^x u(\eta, t) \text{d}\eta \right] \text{d}W(t) & \text{in } Q, \\
    \text{d}\dot{u} + u_{xxxx} \text{d}t = \left[ \tilde{E}_1(u, \dot{u}) + \tilde{F}_2(u, \dot{u}) \right] \text{d}t \\
    \quad + \left[ c_4 u + c_4 x \int_0^x u(\eta, t) \text{d}\eta \right] \text{d}W(t) & \text{in } Q, \\
    u(0, t) = u_x(0, t) = 0 & \text{on } (0, T), \\
    u(1, t) = u_x(1, t) = 0 & \text{on } (0, T), \\
    u(x, 0) = h_{0,x}(x), \quad \dot{u}(x, 0) = \dot{h}_{0,x}(x) & \text{in } (0, 1),
\end{cases}
\tag{5.3}
\]

where
\[
\tilde{F}_2(u, \dot{u}) = \left[ c_{2,x} + \left( c_3 \frac{R_x}{R} \right) x - \left( \frac{R_{xxxx}}{R} \right) x \right] \int_0^x u(\eta, t) \text{d}\eta - \left( \frac{R_t}{R} \right) x \int_0^x \dot{u}(\eta, t) \text{d}\eta.
\]

From (5.3), we know that \(u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0\) in \((0, T)\). Applying the Carleman estimate (2.3) to (5.3), there exist \(\mu_1 \geq 1\) and \(\lambda_1 = \lambda_1(\mu) \geq 1\), such that for all \(\mu \geq \mu_1\) and \(\lambda \geq \lambda_1(\mu)\),
\[
\begin{align*}
    &\mathbb{E} \int_Q \theta_2^2 \left[ \lambda^7 \mu^8 \varphi_2^7 u^2 + \lambda^5 \mu^6 \varphi_2^5 u_x^2 + \lambda^3 \mu^4 \varphi_2^3 (u_{xx}^2 + \dot{u}^2) + \lambda \mu^2 \varphi_2 (u_{xxxx}^2 + \dot{u}_x^2) \right] \text{d}x \text{d}t \\
    &\quad \leq C \mathbb{E} \int_Q \theta_2^2 \left[ \lambda^6 \mu^6 \varphi_2^6 \left( \int_0^x u(\eta, t) \text{d}\eta \right)^2 + \lambda^2 \mu^4 \varphi_2^2 \left( \int_0^x u(\eta, t) \text{d}\eta \right)^2 \right] \text{d}x \text{d}t \\
    &\quad + C \mathbb{E} \int_0^T \int_{G_0} \theta_2^2 \left[ (\lambda^7 \mu^8 \varphi_2^7 u^2 + \lambda^5 \mu^6 \varphi_2^5 u_x^2 + \lambda^3 \mu^4 \varphi_2^3 u_{xx}^2 + \lambda \mu^2 \varphi_2 u_{xxxx}^2 + \lambda \mu^2 \varphi_2 u_{xx}^2) \right] \text{d}x \text{d}t.
\end{align*}
\tag{5.4}
\]

Since
\[
\left| \int_0^x u(\eta, t) \text{d}\eta \right|^2 \leq \int_0^1 |u(\eta, t)|^2 \text{d}\eta,
\]
we see
\[
\int_Q \left| \int_0^x u(\eta, t) \text{d}\eta \right|^2 \text{d}x \text{d}t \leq \int_Q |u(\eta, t)|^2 \text{d}x \text{d}t.
\]

Combing the above inequality with (5.4), there exist \(\mu_2 > 0\) and \(\lambda_2 = \lambda_2(\mu)\), such that for all \(\mu \geq \mu_2\) and \(\lambda \geq \lambda_2(\mu)\),
\[
\begin{align*}
    &\mathbb{E} \int_Q \theta_2^2 \left[ \lambda^7 \mu^8 \varphi_2^7 u^2 + \lambda^5 \mu^6 \varphi_2^5 u_x^2 + \lambda^3 \mu^4 \varphi_2^3 (u_{xx}^2 + \dot{u}^2) + \lambda \mu^2 \varphi_2 (u_{xxxx}^2 + \dot{u}_x^2) \right] \text{d}x \text{d}t \\
    &\quad \leq C \mathbb{E} \int_Q \theta_2^2 \left[ (\lambda^7 \mu^8 \varphi_2^7 u^2 + \lambda^5 \mu^6 \varphi_2^5 u_x^2 + \lambda^3 \mu^4 \varphi_2^3 u_{xx}^2 + \lambda \mu^2 \varphi_2 u_{xxxx}^2 + \lambda \mu^2 \varphi_2 u_{xx}^2) \right] \text{d}x \text{d}t.
\end{align*}
\]
Together the above inequality with \( y = 0 \) in \( G_0 \times (0, T) \) implies that
\[
u = 0 \text{ in } G_0 \times (0, T), \; \mathbb{P}\text{-a.s.,}
\]
we have
\[
u = \hat{\nu} = 0 \text{ in } Q, \; \mathbb{P}\text{-a.s.},
\]
and hence,
\[
y = \hat{y} = 0 \text{ in } Q, \; \mathbb{P}\text{-a.s.},
\]
which means
\[
A(\cdot) = B(\cdot) = H(\cdot) = P(\cdot) = 0 \text{ in } (0, T), \; \mathbb{P}\text{-a.s.}
\]
\[\square\]

6 Summary

This paper considers the global Carleman estimates of refined stochastic beam equations with the uncertainty between the velocity and displacement of the beam model. By establishing a fundamental weighted identity and properly choosing weight functions, two Carleman estimates are established, with which the exact controllability of the refined system is proved, and meanwhile, the uniqueness of an inverse drift source problem is obtained without any requirement on the initial and terminal values.

It will be interesting to further consider the following problems:
(i) Whether one of boundary controls can be removed in the controllability result.
(ii) Whether the refined system is exactly controllable when control \( g \in L^2_F(0, T; L^2(0, 1)) \).
(iii) How to get the quantitative estimates for the inverse diffusion source problem.
(iv) How to obtain inverse source result by boundary observation or internal observation alone.

Appendix A. Proof of Lemma 3.1

Proof of Lemma 3.1. By (3.1) and \( z = \theta y, \; \hat{z} = \theta \hat{y} + \ell_t z \), we obtain that
\[
dz = d(\theta y) = \theta \ell_t y dt + \theta dy = \hat{z} dt + \theta f_1 dt + \theta g_1 dW(t).
\]
(6.1)

Hence,
\[
d\hat{y} = d[\theta^{-1}(\hat{z} - \ell_t z)] = \theta^{-1}[d\hat{z} - \ell_t \hat{z} dt - \ell_t dz - \ell_t(\hat{z} - \ell_t z)dt]
\]
\[
= \theta^{-1}[d\hat{z} - 2\ell_t \hat{\hat{z}} dt + (\ell_t^2 - \ell_t \ell_t)z dt - \theta \ell_t f_1 dt - \theta \ell_t g_1 dW(t)].
\]

Recalling \( y = \theta^{-1}z \), it holds that
\[
yxxxx = \theta^{-1}[zxxxx - 4\ell_x zxxx + 6(\ell_x^2 - \ell_x) zxx - 4Dz_x
\]
\[+ (\ell_x^4 - 6\ell_x^2 \ell_x + 3\ell_x^2 + 4\ell_x \ell_x - \ell_x)z].
\]
Therefore,

\[ \theta(d\hat{y} + y_{xxxx}dt) \]
\[ = \hat{d}z + x_{xxxx}dt - 2\ell_1\hat{z}dt - 4\ell_xz_{xx}dt + 6(\ell_x^2 - \ell_{xx})z_{xx}dt - 4Dz_xdt \]
\[ + (\ell_x^2 - 6\ell_x^2\ell_{xx} + 3\ell_x^2 + 4\ell_x\ell_{xxx} - \ell_{xxxx} + \ell_x^2 - \ell_{tt})zdt - \theta\ell_1f_1dt - \theta\ell_1g_1dW(t). \]

From the definitions of \( \Phi \) and \( \Psi \), one can get that

\[ \theta I(d\hat{y} + y_{xxxx}dt) = I^2 + I[d\hat{z} + x_{xxxx}dt + 6(\ell_x^2 - \ell_{xx})z_{xx}dt + (\Phi - \Psi)zdt] \]
\[ - \theta\ell_1f_1dt - \theta\ell_1g_1dW(t) \]
\[ = I_1 + I_2 + I_3 + I_4 + I^2 - \theta\ell_1f_1dt - \theta\ell_1g_1dW(t), \]

where

\[ I_1 = I\hat{d}z, \quad I_2 = Ix_{xxxx}dt, \quad I_3 = 6I(\ell_x^2 - \ell_{xx})z_{xx}dt, \quad I_4 = I(\Phi - \Psi)zdt. \]

Now, we analyze the right-hand side of (6.2). For the first one, we have

\[ I_1 = d(-\ell_t\hat{z}^2 - 4\ell_xz_{xxx}\hat{z} - 4Dz_x\hat{z} + \Psi z\hat{z}) + \ell_t\hat{z}^2dt + \ell_t(d\hat{z})^2 \]
\[ + 4\ell_xz_{xxx}\hat{z}dt + 4\ell_x\hat{d}zd_{xxx} + 4\ell_xd\hat{z}d_{xxx} + 4Dz_x\hat{z}dt \]
\[ + 4Dz_d\hat{d}zd_x + 4Dd\hat{z}dz_x - \Psi z\hat{z}dt - \Psi \hat{z}dz - \Psi d\hat{z}dz. \]

Because of \( dz = \hat{z}dt + \theta f_1dt + \theta g_1dW(t) \), it follows that

\[ 4\ell_x\hat{z}d_{xxx} + 4D\hat{z}dz_x - \Psi \hat{z}dz \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)]_x \]
\[ + 4D\hat{z}d[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)] - \Psi \hat{z}d[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)]_x \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)]_x \]
\[ + 4D\hat{z}[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)]_x - \Psi \hat{z}[\hat{z}dt + \theta f_1dt + \theta g_1dW(t)]_x \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 2(2\ell_x\hat{z}_x + \ell_x^2\hat{z}^2 - D\hat{z}^2 - \ell_{xxxx}\hat{z}^2)_x + 6\ell_x\hat{z}_x^2dt \]
\[ + (2\ell_x\hat{z}_x + 2D_x + \Psi)\hat{z}^2 dt - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta f_1)_x dt + 4D\hat{z}(\theta f_1)_x dt - \Psi \hat{z}\theta f_1 dt - \Psi \hat{z}\theta f_1 dt \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 2(2\ell_x\hat{z}_x + \ell_x^2\hat{z}^2 - D\hat{z}^2 - \ell_{xxxx}\hat{z}^2)_x dt \]
\[ + (2\ell_x\hat{z}_x + 2D_x + \Psi)\hat{z}^2 dt - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta f_1)_x dt + 4D\hat{z}(\theta f_1)_x dt - \Psi \hat{z}\theta f_1 dt - \Psi \hat{z}\theta f_1 dt \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 2(2\ell_x\hat{z}_x + \ell_x^2\hat{z}^2 - D\hat{z}^2 - \ell_{xxxx}\hat{z}^2)_x dt \]
\[ + (2\ell_x\hat{z}_x + 2D_x + \Psi)\hat{z}^2 dt - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta f_1)_x dt + 4D\hat{z}(\theta f_1)_x dt - \Psi \hat{z}\theta f_1 dt - \Psi \hat{z}\theta f_1 dt \]
\[ = 4(\ell_x\hat{z}d_{xxx})_x - 2(2\ell_x\hat{z}_x + \ell_x^2\hat{z}^2 - D\hat{z}^2 - \ell_{xxxx}\hat{z}^2)_x dt \]
\[ + (2\ell_x\hat{z}_x + 2D_x + \Psi)\hat{z}^2 dt - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta f_1)_x dt + 4D\hat{z}(\theta f_1)_x dt - \Psi \hat{z}\theta f_1 dt - \Psi \hat{z}\theta f_1 dt \].

So,

\[ I_1 = d(-\ell_t\hat{z}^2 - 4\ell_xz_{xxx}\hat{z} - 4Dz_x\hat{z} + \Psi z\hat{z}) + 4(\ell_x\hat{z}d_{xxx})_x - (4\ell_x\hat{z}_x + 2\ell_x^2\hat{z}_x) \]
\[ - 2D\hat{z}^2 - 2\ell_{xxxx}\hat{z}^2) _dt + (\ell_{tt} - 2\ell_{xxxx} - 2D_x - \Psi)\hat{z}^2 dt + 6\ell_x\hat{z}_x^2 dt - \Psi \hat{z}d\hat{z} dt \]
\[ + 4D_t\hat{z}dz_x dt + 4\ell_xd\hat{z}d_{xxx} dt - 4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta f_1)_x dt + 4D\hat{z}(\theta f_1)_x dt \]
\[ - \Psi \hat{z}\theta f_1 dt + \ell_t(d\hat{z})^2 + 4\ell_xd\hat{z}d_{xxx} + 4Dd\hat{z}dz_x - \Psi dzd\hat{z} \]
\[ - [4(\ell_x\hat{z} + \ell_x\hat{z}_x)(\theta g_1)_x - 4D\hat{z}(\theta g_1)_x + \Psi \hat{z}\theta g_1]dW(t). \]
For the second one, we have

\[ I_2 = (-2\ell_1 z_{xxx} + 2\ell_1 z_{xx}^2 - 4Dz_x z_{xxx} + \Psi z_{xxx})_x dt + 2\ell_1 \dot{z}_x z_{xxx} dt + 2\ell_1 \dot{z}_x z_{xxx} dt + 2\ell_1 \dot{z}_x z_{xxx} dt \\
+ 2\ell_1 z_{xx}^2 \dot{t} - \Psi z_{xxx} dt - \Psi z_{xxx} dt + 4Dz_x z_{xxx} dt + 4Dz_x z_{xxx} dt \\
= (-2\ell_1 z_{xxx} - 2\ell_1 z_{xx}^2 - 4Dz_x z_{xxx} + \Psi z_{xxx} + 2\ell_1 z_{xx} z_{xxx} - \Psi z_{xxx} + 2Dz_x^2 \\
+ 4Dz_x z_{xxx})_x dt + 2\ell_1 \dot{z}_x z_{xxx} dt - 2\ell_1 \dot{z}_x z_{xxx} dt - 2\ell_1 \dot{z}_x z_{xxx} dt + 2\ell_1 z_{xx}^2 \dot{t} \\
- \Psi z_{xxx} dt + \Psi z_{xx}^2 dt + \Psi z_{xxx} dt - 6Dz_x z_{xxx} dt - 4Dz_x z_{xxx} dt. \] (6.4)

By (6.1), it follows that

\[ 2\ell_1 \dot{z}_x z_{xxx} dt = 2\ell_1 z_{xx} [dz - \theta f_1 dt - \theta g_1 dW(t)]_x \\
= 2\ell_1 z_{xx} [dz - \theta f_1 dt - \theta g_1 dW(t)]_x \\
= d(\ell_1 z_{xx}^2)_x dt - \ell_1 (dz_{xx})^2 - 2\ell_1 z_{xx} (\theta f_1)_x dt - 2\ell_1 z_{xx} (\theta g_1)_x dW(t). \]

Therefore,

\[ I_2 = (-2\ell_1 z_{xxx} - 2\ell_1 z_{xx}^2 - 4Dz_x z_{xxx} + \Psi z_{xxx} + 2\ell_1 z_{xx} z_{xxx} - \Psi z_{xxx} + 2Dz_x^2 \\
+ 4Dz_x z_{xxx})_x dt + 2\ell_1 \dot{z}_x z_{xxx} dt - 2\ell_1 \dot{z}_x z_{xxx} dt + 2\ell_1 z_{xx}^2 \dot{t} - \Psi z_{xxx} dt \\
+ (\Psi - 6D_x + \ell_1) z_{xx}^2 dt + (\Psi - 4D_x z_x z_{xx})_x z_{xx} dt - d(\ell_1 z_{xx}^2) + \ell_1 (dz_{xx})^2 \\
+ 2\ell_1 z_{xx} (\theta f_1)_x dt + 2\ell_1 z_{xx} (\theta g_1)_x dW(t). \] (6.5)

For the third one, we have

\[ I_3 = -6[2(\ell_1 \ell_2 - \ell_1 \ell_2)_x z_{xx} + 2(\ell_2^3 - \ell_1 \ell_2) z_{xx} - (\Psi^2_x - \Psi \ell_x) z_{xx} \\
+ 2D(\ell_2^2 - \ell_2 \ell_2)_x z_{xx} + 12(\ell_2^3 - \ell_1 \ell_2) \dot{z}_{xx} dt + 12(\ell_2^3 - \ell_1 \ell_2) \dot{z}_{xx} dt \\
+ 12(\ell_2^3 - \ell_1 \ell_2)_x z_{xx}^2 dt - 6(\Psi^2_x - \Psi \ell_x) z_{xx}^2 dt + 12[D(\ell_2^2 - \ell_1 \ell_2)_x]_x z_{xx}^2 dt \\
- 6(\Psi^2_x - \Psi \ell_x)_x z_{xx}^2 dt \\
= -3[4(\ell_1 \ell_2^2 - \ell_1 \ell_2) z_{xx} + 4(\ell_1 \ell_2^2 - \ell_1 \ell_2) z_{xx} - 2(\Psi^2_x - \Psi \ell_x) z_{xx} \\
+ 4D(\ell_2^2 - \ell_2 \ell_2)_x z_{xx}^2 + (\Psi^2_x - \Psi \ell_x)_x z_{xx}^2 dt + 12(\ell_2^3 - \ell_1 \ell_2)_x \dot{z}_{xx} dt \\
+ 12(\ell_2^3 - \ell_1 \ell_2)_x z_{xx} \dot{z}_{xx} dt + 12(\ell_2^3 - \ell_1 \ell_2)_x z_{xx}^2 dt + 3(\Psi^2_x - \Psi \ell_x)_x z_{xx}^2 dt \\
+ 12[D(\ell_2^2 - \ell_1 \ell_2)]_x z_{xx}^2 dt - 6(\Psi^2_x - \Psi \ell_x)_x z_{xx}^2 dt. \]

By (6.1), it follows that

\[ 12(\ell_1 \ell_2^2 - \ell_1 \ell_2)_x z_{xx} dt \\
= 12(\ell_1 \ell_2^2 - \ell_1 \ell_2)_x [dz - \theta f_1 dt - \theta g_1 dW(t)]_x \\
= 12(\ell_1 \ell_2^2 - \ell_1 \ell_2)_x z_{xx} [dz - \theta f_1 dt - \theta g_1 dW(t)]_x \\
= -12(\ell_1 \ell_2^2 - \ell_1 \ell_2)_x z_{xx} (\theta f_1)_x dt - 12(\ell_1 \ell_2^2 - \ell_1 \ell_2)_x z_{xx} (\theta g_1)_x dW(t). \]
Thus, one can get
\[
I_3 = -3[4(\ell_x^2 - \ell_x \ell_{xx})z^2 + 4(\ell_x^2 - \ell_x \ell_{xx})z^2] + 4D(\ell_x^2 - \ell_x \ell_{xx})z^2 + (\Psi \ell_x^2 - \Psi \ell_{xx})z^2 dt + 6d[(\ell_x^2 - \ell_x \ell_{xx})z^2] + 12(\ell_x^2 - \ell_x \ell_{xx})z^2 dt + 12(\ell_x^2 - \ell_x \ell_{xx})z^2 dt
\]

\[= 12(\ell_x^2 - \ell_x \ell_{xx})(dz_x)^2 + 2(\ell_x^2 - \ell_x \ell_{xx})z_x^2 dt + 12(\ell_x^2 - \ell_x \ell_{xx})z_x^2 dt
\]

For the fourth one, we have
\[
I_4 = -2\ell_t(\Phi - \Psi)z[dz - \theta f_1 dt - \theta g_1 dW(t)] + \Psi(\Phi - \Psi)z^2 dt
\]

\[= d[z - \ell_x(\Phi - \Psi)z^2] - 2[2\ell_x(\Phi - \Psi)z z_{xx} + D(\Phi - \Psi)z^2 - 2(\ell_x \Phi - \ell_x \Psi)z z_{xx}]
\]

\[= -d[(\ell_x \Phi - \ell_x \Psi)z^2] + (\ell_x \Phi - \ell_x \Psi)z z_{xx} + D(\Phi - \Psi)z^2 - 2(\ell_x \Phi - \ell_x \Psi)z z_{xx}]
\]

Together the above equality with (6.2)-(6.6) gives the desired identity (3.2). \qed

Appendix B. Proof of Theorem 2.1

Proof of Theorem 2.1. For \( i = 1, 2 \), recalling the definition of \( \theta_i \), one can see that
\[
\ell_{i,t} = \lambda \eta_{i,t}, \quad \ell_{i,x} = \lambda \mu \varphi_i \psi_{i,x}, \quad \ell_{i,xx} = \lambda \mu^2 \varphi_i \psi_{i,xx}, \quad \ell_{i,xxx} = \lambda \mu^2 \varphi_i \psi_{i,xxx}
\]

and
\[
|\eta_{i,t}| \leq CT e^{2\mu |t|} |\psi_i|, \quad |\varphi_i| \leq CT \varphi_i^3.
\]

Notice that \( y(0, t) = y(1, t) = 0 \) on \( (0, T) \), and
\[
f_1 \in L^2(T; H^0_0(0, 1)), g_1 \in L^2(T; H^4(0, 1) \cap H^0_0(0, 1)).
\]

Then,
\[
\dot{y}(0, t) = \dot{y}(1, t) = \dot{y}_x(0, t) = \dot{y}_x(1, t) = 0 \quad \text{on} \quad (0, T).
\]
Choosing $\theta = \theta_1$ and $\ell = \ell_1$ in (3.2). Then, by integrating (3.2) in $Q$ and taking expectation, noting that $\theta_1(0) = \theta_1(T) = 0$, we have

$$
\mathbb{E} \int_Q \theta_1 I(d\hat{y} + y_{xxx} dt) dx
= \mathbb{E} \int_Q \left[ V_x dt + 4(\ell_{1,x} \hat{z} dz_{xx})_x \right] dx + \mathbb{E} \int_Q \left\{ I^2 + Az^2 + Bz_x^2 + 6\ell_{1,xx} z_{xx}^2 
+ \left[ \Psi - 6D_x + \ell_{1,tt} + 12(\ell^3_{1,x} - \ell_{1,x} \ell_{1,xx}) \right] z_{xx}^2 + 2\ell_{1,xx} z_{xxx}^2 + P + U \right\} dx dt + \mathbb{E} \int_Q \left[ \ell_{1,t}(\Phi - \Psi)(dz)^2 + \ell_{1,t}(dz_{xx})^2 \right] dx dt
+ \ell_{1,t}(\hat{z})^2 - 6\ell_{1,t}(\ell_{1,x} - \ell_{1,xx})(dz_x)^2 + 4\ell_{1,x} d\hat{z} dz_{xxx} + 4D d\hat{z} dz_x - \Psi d\hat{z} dz \right\} dx.
$$

Now, we evaluate the right-hand side of equality (6.7) term by term. For the first one, by recalling the definition of $V$, and noting that $\psi_{1,x} < 0$ in $[0,1]$, we have

$$
\mathbb{E} \int_Q \left[ V_x dt + 4(\ell_{1,x} \hat{z} dz_{xx})_x \right] dx
= \mathbb{E} \int_Q \left[ -2\ell_{1,x} z_{xxx}^2 + 2D z_{xx}^2 - 12(\ell^3_{1,x} - \ell_{1,x} \ell_{1,xx}) z_{xx}^2 \right] dx dt
= \mathbb{E} \int_0^T \left\{ -2\lambda \mu \varphi_1 \psi_1 z_{xxx}^2 + 6\lambda \mu \varphi_1 \psi_1 \ell_{1,xx}(\lambda \mu \varphi_1 \psi_1^2 + \lambda \mu \varphi_1 \psi_1,xx) - 10\lambda^3 \mu^3 \varphi_1^3 \psi_{1,xx}^2 + 2\lambda \mu \varphi_1 (\mu^2 \psi_1^3 + 3\mu \psi_1,xx + \psi_{1,xxx}) \right\} dx dt
\geq -C\mathbb{E} \int_0^T \left[ \lambda \mu \varphi_1 z_{xxx}^2(0,t) + \lambda^3 \mu^3 \varphi_1^3 z_{xxx}^2(0,t) \right] dt.
$$

For the second one, from the definitions of $A$ and $B$, one can get that

$$
A = \mathcal{O}(\lambda^5 \mu^8 \varphi_1^5) + \mathcal{O}_\mu(\lambda^5 \varphi_1^6) + \mathcal{O}(\lambda^5 \mu^6 \varphi_1^5) - 9\ell^6_{1,x} \ell_{1,xx} + \mathcal{O}(\lambda^6 \mu^8 \varphi_1^6)
+ \mathcal{O}_\mu(\lambda^5 \varphi_1^6) + 2(\ell^7_{1,x})_x + \mathcal{O}(\lambda^6 \mu^8 \varphi_1^6) + \mathcal{O}_\mu(\lambda^5 \varphi_1^6)
= -9\ell^6_{1,x} \ell_{1,xx} + 14\ell^6_{1,x} \ell_{1,xx} + \mathcal{O}(\lambda^6 \mu^8 \varphi_1^6) + \mathcal{O}_\mu(\lambda^5 \varphi_1^6)
= 5\lambda^7 \mu^8 \varphi_1^7 \psi_1^8 + \mathcal{O}(\lambda^7 \mu^7 \varphi_1^7) + \mathcal{O}(\lambda^6 \mu^8 \varphi_1^6) + \mathcal{O}_\mu(\lambda^5 \varphi_1^6)
\geq C\lambda^7 \mu^8 \varphi_1^7,
$$

and

$$
B = 60\ell^6_{1,x} \ell_{1,xx} + \mathcal{O}(\lambda^4 \mu^6 \varphi_1^4) + 54\ell^6_{1,x} \ell_{1,xx} - 6(\ell^5_{1,x})_x + \mathcal{O}(\lambda^4 \mu^6 \varphi_1^4) + \mathcal{O}_\mu(\lambda^3 \varphi_1^4)
= 84\ell^6_{1,x} \ell_{1,xx} + \mathcal{O}(\lambda^4 \mu^6 \varphi_1^4) + \mathcal{O}_\mu(\lambda^3 \varphi_1^4)
= 84\lambda^5 \mu^6 \varphi_1^5 \psi_1^6 + \mathcal{O}(\lambda^5 \mu^5 \varphi_1^5) + \mathcal{O}(\lambda^4 \mu^6 \varphi_1^4) + \mathcal{O}_\mu(\lambda^3 \varphi_1^4)
\geq C\lambda^5 \mu^6 \varphi_1^5,
$$

19
where $O(\mu^k)$ denotes a function of order $\mu^k$ for a sufficiently large $\mu$, and $O_\mu(\lambda^k)$ denotes a function of order $\lambda^k$ for a fixed $\mu$ and sufficiently large $\lambda$. For the third one, by the definitions of $\Psi$ and $D$, we know

$$
\mathbb{E} \int_Q \left[ \Psi - 6D_x + \ell_{1,t} + 12(\ell_{1,x}^3 - \ell_{1,x}\ell_{1,xx}) \right] z_{xx}^2 dxdt
= \mathbb{E} \int_Q \left[ -9\ell_{1,x}^2 + 18\ell_{1,xx} + 6\ell_{1,xxx} - 6\ell_{1,xxxx} + \ell_{1,tt} \right] z_{xx}^2 dxdt
= \mathbb{E} \int_Q \left[ 9\lambda^3 \mu^4 \varphi_1^2 \psi_1 + O(\lambda^3 \mu^3 \varphi_1^3) + O(\lambda \varphi_1^2) + O(\lambda^3 \mu^3 \varphi_1^3) \right] z_{xx}^2 dxdt
\geq C\mathbb{E} \int_Q \lambda^3 \mu^4 \varphi_1^2 z_{xx}^2 dxdt,
$$

and

$$
\mathbb{E} \int_Q (\ell_{1,tt} - 2\ell_{1,xxxx} - 2D_x - \Psi) \dot{z}^2 dxdt
= \mathbb{E} \int_Q \left[ 3\ell_{1,x}^2 + 6\ell_{1,x} + 6\ell_{1,x} \ell_{1,xx} + \ell_{1,tt} - 4\ell_{1,xxxx} \right] \dot{z}^2 dxdt
= \mathbb{E} \int_Q \left[ 3\lambda^3 \mu^4 \varphi_1^2 \psi_1 + O(\lambda^2 \mu^4 \varphi_1^2) + O(\lambda \varphi_1^2) + O(\lambda^3 \mu^3 \varphi_1^3) \right] \dot{z}^2 dxdt
\geq C\mathbb{E} \int_Q \lambda^3 \mu^4 \varphi_1^2 \dot{z}^2 dxdt.
$$

For the fourth one, by Itô formula, one can obtain that

$$
\mathbb{E} \int_Q \left[ \ell_{1,t}(\Phi - \Psi)(dz)^2 - 6\ell_{1,t}(\ell_{1,x}^2 - \ell_{1,xx})(dz_x)^2 + \ell_{1,t}(dz_{xx})^2 + \ell_{1,t}(d\dot{z})^2
+ 4\ell_{1,x} d\dot{z} z_{xxx} + 4Dd\dot{z} z_x - \Psi dz \dot{z} d\dot{z} \right] dx
= \mathbb{E} \int_Q \left\{ \ell_{1,t}(\Phi - \Psi)(\theta_1 g_1)^2 - 6\ell_{1,t}(\ell_{1,x}^2 - \ell_{1,xx})(\theta_1 g_1_x)^2 + \ell_{1,t}(\theta_1 g_1_{xx})^2
+ \theta_1^2 \ell_{1,t}(\ell_{1,t} g_1 + g_2)^2 + 4\theta_1 D(\ell_{1,t} g_1 + g_2)(\theta_1 g_1)_x
+ 4\theta_1 \ell_{1,x}(\ell_{1,t} g_1 + g_2)(\theta_1 g_1)_{xxx} - \Psi \theta_1^2 g_1(\ell_{1,t} g_1 + g_2) \right\} dx dt
\geq -C\mathbb{E} \int_Q \left[ \ell_{1,t}((\ell_{1,x}^2 g_1 + \ell_{1,xx}^2 g_1 + g_1_{xx}^2 + \ell_{1,t} g_1 + g_2 + |\ell_{1,x}^3 g_1 g_1_{xx}| + |\ell_{1,x} g_1 g_1_{xxx}| + |\ell_{1,x} g_1 g_1_{xx} g_2| + \ell_{1,x}^4 |g_1 g_2|) \right] dx dt
\geq -C\mathbb{E} \int_Q \theta_1^2 (\lambda^6 \varphi_1^6 g_1 + \lambda^4 \varphi_1^4 g_1^2 + \lambda^2 \varphi_1^2 g_1_{xx} + g_1_{xxx} + \lambda^6 \varphi_1^6 g_2^2) dx dt.
$$

For the fifth one, from the definitions of $U$ and $P$, there exist $\lambda_1, \mu_1 \geq 1$ such that for any
\( \lambda \geq \lambda_1 \) and \( \mu \geq \mu_1 \), one can get

\[
\left| E \int Q U \, dx dt \right| \\
\leq C E \int_Q \theta_1 \left[ \left| \ell_{1,t} \right| (|\ell_{1,t}^3 f_1| + |\ell_{1,x}^3 z_{1xx} f_1| + |\ell_{1,x}^3 z_{1xx} f_1| + \ell_{1,x}^4 z f_1| + |\ell_{1,x}^4 z f_1| + |\ell_{1,x}^5 z f_1| + |\ell_{1,x}^5 z f_1| + |\ell_{1,x}^6 z f_1| + |\ell_{1,x}^6 z f_1|) \right] dx dt
\]

\[\leq C E \int_Q \theta_1^2 (\lambda^6 \mu^6 \psi_1^6 z_1^2 + \lambda^4 \mu^4 \psi_1^4 f_{1,x}^2 + \lambda^2 \mu^2 \psi_1^2 f_{1,xx}^2) dx dt
\]

\[
+ C E \int_Q \left( \lambda^4 \mu^4 \psi_1^4 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 \right) dx dt,
\]

and

\[
\left| E \int Q P \, dx dt \right| \\
\leq C E \int_Q \left( \lambda \mu \psi_1^2 z_{1x} + \lambda \mu \psi_1^2 z_{1xx} + \lambda \mu \psi_1^2 z_{1xx} \right) dx dt
\]

\[\leq C E \int_Q \left( \lambda^3 \mu^3 \psi_1^3 z_{1x} + \lambda^3 \mu^3 \psi_1^3 z_{1xx} \right) dx dt
\]

\[\leq C E \int_Q \left[ \lambda^6 \mu^6 \psi_1^6 z_1^2 + \lambda^4 \mu^4 \psi_1^4 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 + \lambda^2 \mu^2 \psi_1^2 z_1^2 \right] dx dt.
\]

By (6.7)-(6.15), there exist two sufficiently large \( \lambda_2, \mu_2 \geq 1 \), such that for any \( \lambda \geq \lambda_2 \) and \( \mu \geq \mu_2 \), it holds that

\[
E \int Q \left[ \lambda^7 \mu^7 \psi_1^7 z_1^2 + \lambda^5 \mu^5 \psi_1^5 z_1^2 + \lambda^3 \mu^3 \psi_1^3 (z_{1xx}^2 + \hat{z}_x^2) + \lambda \mu \psi_1 (z_{1xx}^2 + \hat{z}_x^2) \right] dx dt
\]

\[\leq C E \int_Q \theta_1^2 (\lambda^6 \mu^6 \psi_1^6 f_1^2 + \lambda^4 \mu^4 \psi_1^4 f_{1,xx}^2 + \lambda^2 \mu^2 \psi_1^2 f_{1,xxx}^2 + f_2^2
\]

\[+ \lambda^6 \psi_1^6 g_1^2 + \lambda^4 \psi_1^4 g_{1,xx}^2 + \lambda^2 \psi_1^2 g_{1,xxx}^2 + \lambda \mu \psi_1 (g_{1,xxx}^2 + \hat{g}_{1,xxx}^2) dx dt
\]

\[+ C E \int_0^T \left[ \lambda \mu \psi_1 z_{1xxx}^2(0,t) + \lambda^3 \mu^3 \psi_1^3 z_{1xxx}^2(0,t) \right] dt.
\]

From

\[y_x = \theta_1^{-1}(z_x - \ell_{1,x} z) = \theta_1^{-1}(z_x - \lambda \mu \varphi_1 \psi_1),\]

and

\[z_x = \theta_1(y_x + \ell_{1,x} y) = \theta_1(y_x + \lambda \mu \varphi_1 \psi_1),\]

we get that

\[
\frac{1}{C \theta_1^2} (y_x^2 + \lambda^2 \mu^2 \varphi_1^2 y_x^2) \leq z_x^2 + \lambda^2 \mu^2 \varphi_1^2 z_x^2 \leq C \theta_1^2 (y_x^2 + \lambda^2 \mu^2 \varphi_1^2 y_x^2).
\]
Likewise, $z_{xx}$ and $y_{xx}$, $z_{xxx}$ and $y_{xxx}$, $\dot{z}$ and $\dot{y}$, $\dot{z}_x$ and $\dot{y}_x$ have similar estimates. Therefore, there exist $\mu_3 > 0$ and $\lambda_3 = \lambda_3(\mu)$, such that for any $\mu \geq \mu_3$ and $\lambda \geq \lambda_3(\mu)$, it follows that

$$\mathbb{E} \int_0^T \left[ \lambda^7 \mu^8 \varphi^2_1 y^2 + \lambda^5 \mu^6 \varphi^5_1 y^2 + \lambda^3 \mu^4 \varphi^3_1 (y^2_x + \dot{y}^2) + \lambda \mu^2 \varphi_1 (y^2_{xxx} + \dot{y}^2_x) \right] dt$$

$$\leq C \mathbb{E} \int_0^T \left[ \lambda^7 \mu^8 \varphi^2_1 f^2 + \lambda^6 \mu^6 \varphi^2_1 f^2 + \lambda^4 \mu^4 \varphi^2_1 f^2 + \lambda^2 \mu^2 \varphi^2_1 f^2 + \lambda \mu \varphi_1 f^2 \right] dt + C \mathbb{E} \int_0^T \left[ \lambda^3 \mu^3 \varphi^3_1 z^2 \right] dt.$$

Thus, the desired result (2.2) in Theorem 2.1 is obtained. □

**Appendix C. Proof of Theorem 2.2**

**Proof of Theorem 2.2.** Taking $\theta = \theta_2$ and $\ell = \ell_2$ in Lemma 3.1. Note that $\psi_{2,x}(0) > 0$ and $\psi_{2,x}(1) < 0$, then, by integrating (3.2) on $Q$ and taking expectation, for sufficiently large $\lambda$ and $\mu$, we have

$$\mathbb{E} \int_Q \left[ V_x dt + 4(\ell_{2,x} \dot{z}_x) dx \right] dx$$

$$= \mathbb{E} \int_0^T \left\{ -2 \lambda \mu \varphi_2 \psi_{2,x} z^2_{xxx} + \left[ 6 \lambda \mu \varphi_2 \psi_{2,x} \ell_{2,x,x} (\lambda \mu \varphi_2 \psi^2_{2,x} + \lambda \mu \varphi_2 \psi_{2,xx}) \right. \right.$$

$$\left. -10 \lambda^3 \mu^3 \varphi^3_2 \psi^3_{2,x} + 2 \lambda \mu \varphi_2 (\mu^2 \psi^2_{2,x} + 3 \mu \psi_{2,x} \psi_{2,xx} + \psi_{2,xxx}) \right] z^2_x \right\} dt \quad (6.16)$$

$$= \mathbb{E} \int_0^T \left\{ -2 \lambda \mu \varphi_2 \psi_{2,x} z^2_{xxx} + \left[ -2 \lambda^3 \mu^3 \varphi^3_2 \psi^3_{2,x} + O(\lambda^2 \mu^2 \varphi^2_2) \right] z^2_x \right\} dt$$

$$\geq 0.$$

Similar to the derivation of (6.9)-(6.15) and combing with (6.16), there are two positive constants $\lambda_4, \mu_4 \geq 1$ such that for any $\lambda \geq \lambda_4$ and $\mu \geq \mu_4$, it holds that

$$\mathbb{E} \int_Q \left[ \lambda^7 \mu^8 \varphi^2_2 z^2 + \lambda^5 \mu^6 \varphi^5_2 z^2 + \lambda^3 \mu^4 \varphi^3_2 (z^2_{xx} + \dot{z}^2) + \lambda \mu^2 \varphi_2 (z^2_{xxx} + \dot{z}^2_x) \right] dt$$

$$\leq C \mathbb{E} \int_Q \left[ \lambda^7 \mu^8 \varphi^2_2 f^2 + \lambda^6 \mu^6 \varphi^2_2 f^2 + \lambda^4 \mu^4 \varphi^2_2 f^2 + \lambda^2 \mu^2 \varphi^2_2 f^2 + \lambda^4 \varphi^2_2 g^2 \right] dt + C \mathbb{E} \int_0^T \left[ \lambda^7 \mu^8 \varphi^2_2 z^2 + \lambda^5 \mu^6 \varphi^5_2 z^2 + \lambda \mu^2 \varphi_2 (z^2_{xxx} + \dot{z}^2_x) \right] dt.$$

By $z = \theta_2 y$, we have

$$\frac{1}{C} \theta^2_2 (y^2_x + \lambda \mu^2 \varphi^2_{2,y}) \leq \frac{z^2_x}{C} + \lambda \mu^2 \varphi^2_{2,y} \leq C \theta^2_2 (y^2_x + \lambda \mu^2 \varphi^2_{2,y}).$$
Likewise, $z_{xx}$ and $y_{xx}$, $z_{xxx}$ and $y_{xxx}$, $\dot{z}$ and $\dot{y}$, $\dot{z}_x$ and $\dot{y}_x$ have similar estimates. Therefore, there exist $\mu_5 > 0$ and $\lambda_5 = \lambda_5(\mu)$, such that for any $\mu \geq \mu_5$ and $\lambda \geq \lambda_5$, it follows that

$$
\mathbb{E} \int \theta_2^2 \left[ \lambda^7 \mu^8 \phi^7 y^2 + \lambda^5 \mu^6 \phi^5 y_{xx}^2 + \lambda^4 \mu^4 \phi^3 y_{xxx}^2 + \lambda^2 \mu^2 \phi^2 y_{xxxx}^2 + \lambda \mu \phi y_{xxxxxx}^2 \right] dx dt
$$

$$
\leq C \mathbb{E} \int_Q \theta_2^2 \left( \lambda^6 \mu^6 \phi^4 y_{f1}^2 + \lambda^4 \mu^4 \phi^2 y_{f2}^2 + \lambda^2 \mu^2 \phi^2 y_{f1,xx}^2 + \lambda \mu \phi y_{f1,xxx}^2 \right) dx dt
$$

$$
+ \lambda^2 \phi^2 y_{f1,xxxx}^2 + \lambda^0 \phi^0 y_{f1,xxxxxx}^2 + \lambda^2 \mu^2 \phi^2 y_{f1,xxx}^2 + \lambda \mu \phi y_{f1,xxxxxx}^2 \right) dx dt + C \mathbb{E} \int_0^T \int_{G_1} \theta_2^2 \left[ \lambda^7 \mu^8 \phi^7 y^2 
$$

$$
+ \lambda^5 \mu^6 \phi^5 y_{xx}^2 + \lambda^4 \mu^4 \phi^3 y_{xxx}^2 + \lambda^2 \mu^2 \phi^2 y_{xxxx}^2 + \lambda \mu \phi y_{xxxxxx}^2 \right] dx dt.
$$

(6.17)

Next, we give the estimates on $\mathbb{E} \int_0^T \int_{G_1} \theta_2^2 \lambda^3 \mu^4 \phi^3 \tilde{\zeta} y^2 dx dt$ and $\mathbb{E} \int_0^T \int_{G_1} \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{y}^2 dx dt$. To this aim, choose a function $\tilde{\zeta} \in C_0^\infty (G_0)$ satisfying $0 \leq \tilde{\zeta} \leq 1$ in $G_0$, $\tilde{\zeta} \equiv 1$ in $G_1$. Note that

$$
d(\theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} y \tilde{y}) = \theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} (y \tilde{y} + \tilde{y} dy + dy \tilde{y}) + (\theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta})_t \tilde{y} dt.
$$

Then, by a simple calculation and Hölder’s inequality, we have that for any $\rho > 0$,

$$
\mathbb{E} \int_Q \theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} \tilde{y}^2 dx dt
$$

$$
= \mathbb{E} \int_Q \left[ \theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} y_{xx}^2 + (\theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta})_t y_{xx} y_{xxx} - (\theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta})_t y_{xxx} \right. 
$$

$$
- \theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} (y f_2 + \tilde{y} f_1 + g_1 g_2) - (\theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta})_t \tilde{y} \right] dx dt
$$

(6.18)

$$
\leq C \mathbb{E} \int_Q \theta_2^2 \tilde{\zeta} (\lambda^6 \mu^6 \phi^6 y_{xx}^2 + \lambda^5 \mu^5 \phi^5 y_{x}^2 + \lambda^4 \mu^4 \phi^4 y_{xxx}^2 + \lambda^3 \mu^3 \phi^3 y_{xxxx}^2 + \lambda^2 \mu^2 \phi^2 y_{xxxxxx}^2 + \lambda \mu \phi y_{xxxxxx}^2 
$$

$$
+ \lambda^2 \mu^2 \phi^2 y_{f1}^2 + \lambda^6 \mu^6 \phi^6 y_{f1}^2 + \lambda^4 \mu^4 \phi^2 y_{f2}^2 + \lambda^2 \mu^2 \phi^2 y_{f1,xx}^2 + \lambda \mu \phi y_{f1,xxx}^2 \right) dx dt + \rho \mathbb{E} \int_Q \theta_2^2 \lambda^3 \mu^4 \phi^2 \tilde{\zeta} \tilde{y}^2 dx dt.
$$

Similarly, by $d(\theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} y_{xx} \tilde{y})$ and Itô’s formula, it can be seen that for any $\rho > 0$,

$$
\mathbb{E} \int_Q \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} \tilde{y}^2 dx dt = \mathbb{E} \int_Q \left[ \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} y_{xx}^2 + (\theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta})_t y_{xx} y_{xxx} \right. 
$$

$$
- \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} (y f_{xx} + \tilde{y} f_{xx} + g_{xx}) + g_{xx} g_{xx} - \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} y_{xxx} \right] dx dt
$$

(6.19)

$$
\leq C \mathbb{E} \int_Q \theta_2^2 \tilde{\zeta} \left[ \lambda^3 \mu^4 \phi^3 y_{xx}^2 + \lambda^2 \mu^2 \phi^2 y_{xxx}^2 + \lambda \mu \phi y_{xxxxxx}^2 \right. 
$$

$$
+ \lambda^2 \mu^2 \phi^2 y_{f1,xx}^2 + \lambda^2 \mu^2 \phi^2 y_{f2}^2 \right] dx dt + \rho \mathbb{E} \int_Q \theta_2^2 \lambda^2 \mu^2 \phi^2 \tilde{\zeta} \tilde{y}^2 dx dt.
$$

Finally, combing (6.17)-(6.19), one can get the desired estimate (2.3).
References


