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NEAR OPTIMALITY OF STOCHASTIC CONTROL FOR SINGULARLY PERTURBED MCKEAN–VLASOV SYSTEMS*

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Abstract. In this paper, we are concerned with the optimal control problems for a class of systems with fast-slow processes. The problem under consideration is to minimize a functional subject to a system described by a two-time scaled McKean–Vlasov stochastic differential equation whose coefficients depend on state components and their probability distributions. Firstly, we establish the existence and uniqueness of the invariant probability measure for the fast process. Then, by using the relaxed control representation and the martingale method, we prove the weak convergence of the slow process and control process in the original problem, and we obtain an associated limit problem in which the coefficients are determined by the average of those of the original problem with respect to the invariant probability measure. Finally, by establishing the nearly optimal control of the limit problem, we obtain the near optimality of the original problem.

Key words. stochastic control, McKean–Vlasov diffusion, singular perturbation, weak convergence, martingale method, near optimality

MSC codes. 93E20, 93C70, 60H10, 60G57

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1. Introduction.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $W^1(t)$ and $W^2(t)$ are two mutually independent d_1 and d_2-dimensional standard Brownian motions defined on this space. For any $d \geq 1$, $\mathcal{P}(\mathbb{R}^d)$ denotes the set of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and $\mathcal{P}_2(\mathbb{R}^d)$ denotes the set of the elements in $\mathcal{P}(\mathbb{R}^d)$ with finite second moment.

In this paper, we consider the controlled system which can be formulated as the following two-time scaled McKean–Vlasov stochastic differential equation (SDE):

(1.1)
$$\begin{cases} dX^\varepsilon(t) = \dfrac{1}{\varepsilon} b(X^\varepsilon(t), \mathscr{L}(X^\varepsilon(t)))dt + \dfrac{1}{\sqrt{\varepsilon}}\sigma(X^\varepsilon(t), \mathscr{L}(X^\varepsilon(t)))dW^1(t), X^\varepsilon(0) = \xi, \\ dY^\varepsilon(t) = f(X^\varepsilon(t), \mathscr{L}(X^\varepsilon(t)), Y^\varepsilon(t), \mathscr{L}(Y^\varepsilon(t)), u^\varepsilon(t))dt \\ \qquad\qquad + g(X^\varepsilon(t), \mathscr{L}(X^\varepsilon(t)), Y^\varepsilon(t), \mathscr{L}(Y^\varepsilon(t)))dW^2(t), \qquad\qquad Y^\varepsilon(0) = \zeta, \end{cases}$$

where the small parameter $\varepsilon > 0$ represents the ratio between the time scale of $X^\varepsilon(t)$ and $Y^\varepsilon(t)$; $\mathscr{L}(X^\varepsilon(t))$ and $\mathscr{L}(Y^\varepsilon(t))$ denote the probability distributions of $X^\varepsilon(t)$ and $Y^\varepsilon(t)$, respectively; $b : \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \to \mathbb{R}^{d_1}$, $\sigma : \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \to \mathbb{R}^{d_1\times d_1}$, $f : \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \times U \to \mathbb{R}^{d_2}$, and $g : \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_2\times d_2}$ are Borel measurable functions; and ξ, ζ are two $\mathcal{F}_0$-measurable random variables. In (1.1), $u^\varepsilon(t)$ is called the *control process* valued in a compact set $U \subset \mathbb{R}^r$.

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Let the running cost function $R : \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \times U \to \mathbb{R}$ and the terminal cost function $Q : \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}$ be measurable, and let $T_0 > 0$ be a given constant. Then, the cost functional is defined as

$$\begin{aligned} J^\varepsilon(u^\varepsilon(\cdot)) = \mathbb{E}\Big[\int_0^{T_0} & R(X^\varepsilon(s), \mathscr{L}(X^\varepsilon(s)), Y^\varepsilon(s), \mathscr{L}(Y^\varepsilon(s)), u^\varepsilon(s))ds \\ & + Q(Y^\varepsilon(T_0), \mathscr{L}(Y^\varepsilon(T_0)))\Big]. \end{aligned} \tag{1.2}$$

A standard optimization problem is to minimize the cost functional $J^\varepsilon(\cdot)$ over an admissible control space $\mathcal{U}^\varepsilon$, that is, to find $u^{*,\varepsilon}(\cdot) \in \mathcal{U}^\varepsilon$ such that

$$J^\varepsilon(u^{*,\varepsilon}(\cdot)) = \inf_{u^\varepsilon(\cdot)\in\mathcal{U}^\varepsilon} J^\varepsilon(u^\varepsilon(\cdot)). \tag{1.3}$$

The motivation for studying the stochastic control problem (1.1)–(1.3), referred to alternatively as the optimal control of singularly perturbed McKean–Vlasov systems or singularly perturbed mean-field stochastic optimal control, comes mainly from various applications of the singularly perturbed stochastic processes in manufacturing systems, finance, economics, control problems, and many other related fields; see [20, 21, 22, 23, 46] and the references therein.

If the system (1.1) and the cost functional (1.2) are independent of the probability distributions $\mathscr{L}(X^\varepsilon(t))$ and $\mathscr{L}(Y^\varepsilon(t))$, then the problem can be reduced to the optimal control problem with classical stochastic system, which has been investigated extensively in the literature. For example, [2] used the dynamic programming approach to study the near optimality of the original problem; [23, 24, 40, 41, 42] established the nearly optimal control by using the weak convergence.

If the system (1.1) and the cost functional (1.2) are distribution-dependent, i.e., the McKean–Vlasov systems, then the corresponding optimal control is a rather new problem. The analysis of McKean–Vlasov SDEs has a long history beginning with the pioneering work [29] and has attracted resurgent attention in recent years thanks to the development of mean-field control, mean-field games, and a wide range of applications in finance, economics, and complex networked systems [3, 8, 14, 26]. So far, McKean–Vlasov SDEs have been investigated considerably on strong and weak well-posedness [8, 15, 16, 17, 27, 39], stochastic control [3, 4, 7, 8, 14, 28, 31, 32, 34, 35, 43, 44], averaging principle for singularly perturbed systems [18, 36], numerical approximation [5, 11], and invariant probability measure [39], among others. However, to the best of our knowledge, there is no result concerning the optimal control of singularly perturbed McKean–Vlasov systems to date.

The optimal control problem considered in this paper is a generalization from classical SDEs [2, 23, 24, 40, 41, 42] to McKean–Vlasov SDEs. Such control problems with singularly perturbed diffusion systems turn out to be rather complicated and difficult to deal with. As a result, it is very important to reduce complexity for computation and analysis. The main idea is as follows: (i) with respect to the original problem, it is shown that there exists a reduced problem (limit problem); then, (ii) by applying the optimal control of the limit problem to the original problem, a nearly optimal control is obtained for the original problem. Compared with [2, 23, 24, 40, 41, 42], the system in the current setting contains the probability distributions of the fast-slow variables. This means that the dynamic programming approach cannot be applied directly.

The objective of this paper is to analyze the near optimality of the optimal control problem (1.1)–(1.3) by using the weak convergence. To proceed, we first prove that the fast process in (1.1) has a unique invariant probability measure and is exponentially ergodic. Then, under suitable assumptions, by proving the weak convergence of the slow process and control process in (1.1), we get a limit problem whose coefficients are determined by the average of those of the original problem (1.1)–(1.3) with respect to the invariant probability measure of the fast process. Roughly, for small ε, the slow-varying equation in (1.1) is close to the following controlled McKean–Vlasov SDE:

$$dY(t) = \bar{f}(Y(t), \mathscr{L}(Y(t)), u(t))dt + \bar{g}(Y(t), \mathscr{L}(Y(t)))dW(t), \quad Y(0) = \zeta, \tag{1.4}$$

and the corresponding cost functional is given by

$$J(u(\cdot)) = \mathbb{E}\Big[\int_0^{T_0} \bar{R}(Y(s), \mathscr{L}(Y(s)), u(s))ds + Q(Y(T_0), \mathscr{L}(Y(T_0)))\Big], \tag{1.5}$$

where $\bar{f}$, $\bar{g}$, and $\bar{R}$ are to be determined in what follows. Note that the limit system (1.4) no longer contains the fast variable and is much simpler than (1.1). Then, in lieu of dealing with the original problem (1.1)–(1.3), we consider the limit problem (1.4)–(1.5). By establishing the optimal or nearly optimal control of the limit problem, we obtain a nearly optimal control of the original problem (1.1)–(1.3).

Compared with the classical control problems of the singularly perturbed systems, the main differences here are to treat the probability distributions of the fast-slow variables in (1.1) and (1.2). To prove the weak convergence of the slow process and control process in the original problem, we use the martingale method and introduce some auxiliary terms to approximate the corresponding distributions. Inspired by [33], we adopt the idea of freezing the slow variable.

The rest of the paper is arranged as follows. In section 2, we introduce some frequently used notation and give the definition of the relaxed controls. In section 3, we formulate the precise problem in terms of the relaxed controls. In section 4, we first deal with the weak convergence and then establish nearly optimal control by virtue of the limit problem. In section 5, we make additional remarks. Finally, Appendix A and Appendix B containing the proofs of two propositions are provided at the end of the paper.

2. Notations and relaxed control. Throughout the paper, unless otherwise specified, we use the following notations. For any $d \geq 1$, let $|\cdot|$ denote the Euclidean norm and $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^d$. For a matrix A, let $\|A\| = \sqrt{\mathrm{tr}[AA^\top]}$ denote the Frobenius norm. For any $p \geq 2$, we consider the following subspace of $\mathcal{P}(\mathbb{R}^d)$:

$$\mathcal{P}_p(\mathbb{R}^d) := \Big\{\mu \in \mathcal{P}(\mathbb{R}^d) : [\mu]_p := \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty\Big\}.$$

Note that for any $x \in \mathbb{R}^d$, the Dirac measure δ_x belongs to $\mathcal{P}_p(\mathbb{R}^d)$. Moreover, $\mathcal{P}_p(\mathbb{R}^d)$ is a Polish space under the L^p-Wasserstein distance

$$W_p(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy)\Big)^{\frac{1}{p}}, \quad \mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^d),$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the collection of all couplings for μ_1 and μ_2. In other words, $\pi \in \mathcal{C}(\mu_1, \mu_2)$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(\cdot \times \mathbb{R}^d) = \mu_1(\cdot)$ and

$\pi(\mathbb{R}^d \times \cdot) = \mu_2(\cdot)$. In particular, if $\mu_1 = \mathscr{L}(X)$ and $\mu_2 = \mathscr{L}(Y)$ are the distributions of random variables X and Y, respectively, then

$$W_p(\mu_1, \mu_2)^p \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \mathscr{L}((X,Y))(dx,dy) = \mathbb{E}|X-Y|^p,$$

where $\mathscr{L}((X,Y))$ represents the distribution of random vector (X,Y); see [6, 8, 37].

Let $C_c^\infty(\mathbb{R}^d;\mathbb{R})$ denote the family of the functions that are infinitely continuously differentiable with compact support; $C([0,T];\mathbb{R}^d)$ ($C([0,\infty);\mathbb{R}^d)$) denote the space of continuous functions on $[0,T]$ ($[0,\infty)$) with values in $\mathbb{R}^d$; C and C_T denote positive constants which may change from place to place, and the subscript T is used to emphasize that the constant depends on T.

Assume that the control space U is a compact set in $\mathbb{R}^r$ and the filtration $\{\mathcal{G}_t^\varepsilon\}_{t\ge 0}$ is given by

$$\mathcal{G}_t^\varepsilon = \sigma\{X^\varepsilon(s), W^1(s), W^2(s), \xi, \zeta; s \le t\}.$$

Then, we introduce the following definition.

Definition 2.1 (see [23, 42]). *A U-valued stochastic process $u(\cdot)$ is called an ordinary admissible control for* (1.1)–(1.3) *if it is $\{\mathcal{G}_t^\varepsilon\}$-progressively measurable and makes the control problem* (1.1)–(1.3) *well-defined. The set of all ordinary admissible controls is denoted by $\mathcal{U}^\varepsilon$. An ordinary admissible control $u(\cdot)$ is said to be a feedback control if there is a U-valued function $u_0(y,\nu)$ such that $u(t) = u_0(Y^\varepsilon(t), \mathscr{L}(Y^\varepsilon(t)))$ for almost all ω, t.*

The use of relaxed controls was initiated in [38] for deterministic systems. Its stochastic counterpart was contained in [13]. Such approaches have regained interest, and such a formulation has been proved to be quite useful for various stochastic control problems; see [23, 24] and the references therein. In what follows, we recall the definition and properties of the relaxed control. Let

$$\begin{aligned}\mathcal{R}(U\times[0,\infty)) = \{m(\cdot) : m(\cdot) &\text{ is a measure on the Borel subsets of } U\times[0,\infty),\\ &\text{and } m(U\times[0,t]) = t\ \forall\ t\ge 0\},\end{aligned}$$

and $\{\mathcal{G}_t\}_{t\ge 0}$ be a given filtration satisfying $\mathcal{G}_t \subseteq \mathcal{F}_t$ (for instance, we take $\mathcal{G}_t = \mathcal{G}_t^\varepsilon$ for the control problem (1.1)–(1.3)).

Definition 2.2 (see [23]). *A random variable $m(\cdot)$ with values in $\mathcal{R}(U\times[0,\infty))$ is said to be an admissible relaxed control if for any $B\in\mathcal{B}(U)$, the function defined by $m(B,t) := m(B\times[0,t])$ is $\{\mathcal{G}_t\}$-progressively measurable. Or equivalently, $m(\cdot)$ is said to be an admissible relaxed control if*

$$\int_0^t \int_B \varphi(u,s) m(duds)$$

is progressively measurable with respect to $\{\mathcal{G}_t\}$ for any $\varphi(\cdot,\cdot)\in C_b(U\times[0,\infty))$ which is the collection of the bounded and continuous functions defined on $U\times[0,\infty)$.

It can be shown that if $m(\cdot)$ is an admissible relaxed control, then there is a measure-valued function $m_t(\cdot)$ (the "derivative") such that $m(dudt) = m_t(du)dt$ and for smooth function $\varphi(\cdot,\cdot)$,

$$\int\int \varphi(u,s)m(duds) = \int ds \int \varphi(u,s) m_s(du).$$

We will be working with the weak convergence of the relaxed control sequence. To proceed, we topologize $\mathcal{R}(U\times[0,\infty))$ as follows. Let $\{\varphi_{n_i} : i<\infty\}$ be a countable dense set of the continuous functions on $U\times[0,n]$ for any $n\geq 1$. Let

$$(m,\varphi)_t=\int_0^t\int_U \varphi(u,s)m(duds),$$

and define

$$d(m_1,m_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}d_n(m_1,m_2),$$

where

$$d_n(m_1,m_2)=\sum_{i=1}^{\infty}\frac{1}{2^i}\Big(\frac{|(m_1-m_2,\varphi_{n_i})_n|}{1+|(m_1-m_2,\varphi_{n_i})_n|}\Big).$$

When we say that $m^n(\cdot)\Rightarrow m(\cdot)$ for a sequence of random measures, we mean the convergence in $\mathcal{R}(U\times[0,\infty))$ under this weak topology; see [42] for more details.

In accordance with [23], any ordinary admissible control $u(\cdot)$ can be represented as a relaxed control by using $m_t(du)=\delta_{u(t)}(du)$, where $\delta_{u(t)}$ is the Dirac measure concentrated at the point $u(t)$. Moreover, we shall establish that any relaxed control can be approximated by the ordinary controls for the distribution-dependent case.

The relaxed control setting has clear advantage since, under such a formulation, the underlying system is linear in the control component. It is thus much easier to obtain limit results and desired optimal controls. For convenience, we define all $m(\cdot)$ on $[0,\infty)$ in what follows. Since the optimization problem considered in this paper is on $[0,T_0]$, we can define the controls or relaxed controls in any admissible way on $[T_0,\infty)$.

3. Problem formulation. In this section, we reformulate the problem in the framework of the relaxed control representation. Following the notations in section 2, an admissible relaxed control for (1.1)–(1.3) is any $\mathcal{R}(U\times[0,\infty))$-valued random variable $m^\varepsilon(\cdot)$, such that for any $B\in\mathcal{B}(U)$, $m^\varepsilon(B,t)$ is progressively measurable with respect to $\{\mathcal{G}_t^\varepsilon\}$, where

$$\mathcal{G}_t^\varepsilon=\sigma\{X^\varepsilon(s),W^1(s),W^2(s),\xi,\zeta;s\leq t\}.$$

Let $\mathcal{R}^\varepsilon$ denote the family of all admissible relaxed controls, that is,

$$\begin{aligned}\mathcal{R}^\varepsilon=\{&m^\varepsilon(\cdot): m^\varepsilon(\cdot)\text{ is }\mathcal{R}(U\times[0,\infty))\text{-valued random variable},\\ &\text{and } m^\varepsilon(B,t)\text{ is }\{\mathcal{G}_t^\varepsilon\}\text{-progressively measurable for any } B\in\mathcal{B}(U)\}.\end{aligned}$$

Owing to the relaxed control formulation, in lieu of (1.1), we consider the following controlled diffusion system:

(3.1)
$$\begin{cases} dX^\varepsilon(t)=\dfrac{1}{\varepsilon}b(X^\varepsilon(t),\mathscr{L}(X^\varepsilon(t)))dt+\dfrac{1}{\sqrt{\varepsilon}}\sigma(X^\varepsilon(t),\mathscr{L}(X^\varepsilon(t)))dW^1(t), \quad X^\varepsilon(0)=\xi,\\ dY^\varepsilon(t)=\Big(\displaystyle\int_U f(X^\varepsilon(t),\mathscr{L}(X^\varepsilon(t)),Y^\varepsilon(t),\mathscr{L}(Y^\varepsilon(t)),u)m_t^\varepsilon(du)\Big)dt\\ \qquad\qquad +g(X^\varepsilon(t),\mathscr{L}(X^\varepsilon(t)),Y^\varepsilon(t),\mathscr{L}(Y^\varepsilon(t)))dW^2(t), \qquad Y^\varepsilon(0)=\zeta,\end{cases}$$

where $m^\varepsilon(\cdot)$ is the relaxed control. Our goal is to find an admissible control $m^{*,\varepsilon}(\cdot) \in \mathcal{R}^\varepsilon$ such that the cost functional

$$\begin{aligned} J^\varepsilon(m^\varepsilon(\cdot)) = \mathbb{E}\Big[\int_0^{T_0} \int_U R(X^\varepsilon(s), \mathscr{L}(X^\varepsilon(s)), Y^\varepsilon(s), \mathscr{L}(Y^\varepsilon(s)), u) m_s^\varepsilon(du)ds \Big] \\ +\mathbb{E}[Q(Y^\varepsilon(T_0), \mathscr{L}(Y^\varepsilon(T_0)))] \end{aligned} \tag{3.2}$$

is minimized. In addition, we define the corresponding optimal value as

$$v^\varepsilon = \inf_{m^\varepsilon(\cdot)\in\mathcal{R}^\varepsilon} J^\varepsilon(m^\varepsilon(\cdot)). \tag{3.3}$$

To proceed, we impose the following assumptions.

(H1) The coefficients b and σ are continuous on $\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$. There exists a positive constant K such that for any $x \in \mathbb{R}^{d_1}$ and $\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})$,

$$|b(x, \mu)| \vee \|\sigma(x, \mu)\| \le K(1 + |x| + W_2(\mu, \delta_0)).$$

(H2) For some $p \ge 2$, there exist positive constants $L_2 > L_1$ such that for any $x_1, x_2 \in \mathbb{R}^{d_1}$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^{d_1})$,

$$\begin{aligned} &2\langle x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2)\rangle + (p-1)\|\sigma(x_1, \mu_1) - \sigma(x_2, \mu_2)\|^2 \\ &\le L_1 W_2(\mu_1, \mu_2)^2 - L_2|x_1 - x_2|^2. \end{aligned}$$

(H3) Assume

$$\begin{aligned} f(x, \mu, y, \nu, u) &= f_1(y, \nu, u) + f_2(x, \mu, y, \nu), \\ R(x, \mu, y, \nu, u) &= R_1(y, \nu, u) + R_2(x, \mu, y, \nu). \end{aligned}$$

The functions f_1, R_1 are continuous on $\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \times U$. Moreover, there exists a positive constant L such that for any $x \in \mathbb{R}^{d_1}$, $y_1, y_2 \in \mathbb{R}^{d_2}$, $u \in U$, $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^{d_1})$, and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2})$,

$$\begin{aligned} &|f_1(y_1, \nu_1, u) - f_1(y_2, \nu_2, u)| \vee |R_1(y_1, \nu_1, u) - R_1(y_2, \nu_2, u)| \\ &\vee|Q(y_1, \nu_1) - Q(y_2, \nu_2)| \le L(|y_1 - y_2| + W_2(\nu_1, \nu_2)) \end{aligned}$$

and

$$\begin{aligned} |f_2(x, \mu_1, y_1, \nu_1) - f_2(x, \mu_2, y_2, \nu_2)| &\vee \|g(x, \mu_1, y_1, \nu_1) - g(x, \mu_2, y_2, \nu_2)\| \\ &\vee |R_2(x, \mu_1, y_1, \nu_1) - R_2(x, \mu_2, y_2, \nu_2)| \\ &\le L(|y_1 - y_2| + W_2(\mu_1, \mu_2) + W_2(\nu_1, \nu_2)). \end{aligned}$$

(H4) There exist constants $\rho, \gamma_1, \gamma_2, \gamma_3 \ge 1$ and K such that for any $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, $u \in U$, $\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})$, and $\nu \in \mathcal{P}_2(\mathbb{R}^{d_2})$,

$$|f_1(y, \nu, u)| \vee |R_1(y, \nu, u)| \le K(1 + |y| + W_2(\nu, \delta_0) + |u|^\rho)$$

and

$$\begin{aligned} |f_2(x, \mu, y, \nu)| &\le K(1 + |x|^{\gamma_1} + |y| + W_2(\mu, \delta_0) + W_2(\nu, \delta_0)), \\ \|g(x, \mu, y, \nu)\| &\le K(1 + |x|^{\gamma_2} + |y| + W_2(\mu, \delta_0) + W_2(\nu, \delta_0)), \\ |R_2(x, \mu, y, \nu)| &\le K(1 + |x|^{\gamma_3} + |y| + W_2(\mu, \delta_0) + W_2(\nu, \delta_0)). \end{aligned}$$

To obtain the desired optimal or nearly optimal control, the properties of the fast process $X^\varepsilon(\cdot)$ in (3.1) are very important. Let $\hat{X}^\varepsilon(t) = X^\varepsilon(\varepsilon t)$. Then, the equation for $\hat{X}^\varepsilon(\cdot)$ can be written as

$$d\hat{X}^\varepsilon(t) = b(\hat{X}^\varepsilon(t), \mathscr{L}(\hat{X}^\varepsilon(t)))dt + \sigma(\hat{X}^\varepsilon(t), \mathscr{L}(\hat{X}^\varepsilon(t)))d\hat{W}^1(t),$$

where $\hat{W}^1(t) = W^1(\varepsilon t)/\sqrt{\varepsilon}$ is a d_1-dimensional Brownian motion. Let $X(t) = \hat{X}^\varepsilon(t)$. Then, $X(\cdot)$ satisfies the following McKean–Vlasov SDE:

$$dX(t) = b(X(t), \mathscr{L}(X(t)))dt + \sigma(X(t), \mathscr{L}(X(t)))d\hat{W}^1(t). \tag{3.4}$$

Note that the existence and uniqueness of the solution to the fast-varying equation in (3.1) for any $\varepsilon > 0$ are equivalent to those of (3.4). The proposition below illustrates the existence and uniqueness of the solution to (3.4). Furthermore, it is shown that (3.4) has a unique invariant probability measure.

Proposition 3.1. *Suppose that assumptions* (H1) *and* (H2) *hold. Then, for any $\mathcal{F}_0$-measurable random variable ξ satisfying $\mathbb{E}|\xi|^p < \infty$,* (3.4) *has a unique strong solution $(X(t))_{t\geq 0}$ with the initial value $X(0) = \xi$, and there exists a positive constant C such that*

$$\sup_{t\geq 0} \mathbb{E}|X(t)|^p \leq C. \tag{3.5}$$

Moreover, (3.4) *has a unique invariant probability measure $\mu \in \mathcal{P}_p(\mathbb{R}^{d_1})$ such that*

$$W_p(P_t^*\mu_0, \mu) \leq W_p(\mu_0, \mu)e^{-\frac{1}{2}(L_2 - L_1)t} \qquad \forall t \geq 0,\ \mu_0 \in \mathcal{P}_p(\mathbb{R}^{d_1}),$$

where $P_t^*\mu_0$ is the distribution of $X(t)$ with the initial condition $\mathscr{L}(X(0)) = \mu_0$ and L_1, L_2 are positive constants given in (H2).

Remark 3.2. According to Proposition 3.1, for any $\varepsilon > 0$ and given initial value ξ satisfying $\mathbb{E}|\xi|^p < \infty$, the fast-varying equation in (3.1) has a unique strong solution $(X^\varepsilon(t))_{t\geq 0}$ and

$$\sup_{0<\varepsilon\leq 1, t\geq 0} \mathbb{E}|X^\varepsilon(t)|^p = \sup_{0<\varepsilon\leq 1, t\geq 0} \mathbb{E}\Big|X\Big(\frac{t}{\varepsilon}\Big)\Big|^p \leq \sup_{t\geq 0} \mathbb{E}|X(t)|^p \leq C.$$

The proof of Proposition 3.1 is presented in Appendix A. For proving the strong well-posedness, the main technique used is the modified Yamada–Watanabe principle [15].

The following proposition states that for any relaxed control $m^\varepsilon(\cdot) \in \mathcal{R}^\varepsilon$, the slow-varying equation in (3.1) has a unique strong solution. Consequently, the system (3.1) is strongly well-posed.

Proposition 3.3. *Let assumptions* (H1)–(H4) *hold, $p \geq 4\max\{\gamma_1, \gamma_2\}$, and the initial condition ξ of the fast-varying equation satisfy $\mathbb{E}|\xi|^p < \infty$. Then, for any $\varepsilon > 0$, admissible relaxed control $m^\varepsilon(\cdot) \in \mathcal{R}^\varepsilon$, and $\mathcal{F}_0$-measurable random variable ζ satisfying $\mathbb{E}|\zeta|^4 < \infty$, the slow-varying equation in* (3.1) *has a unique strong solution $(Y^\varepsilon(t))_{t\geq 0}$ with the initial value $Y^\varepsilon(0) = \zeta$. Moreover, for any $T > 0$, there exists a positive constant C_T such that*

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y^\varepsilon(t)|^4\Big] \leq C_T. \tag{3.6}$$

Remark 3.4. When $p \geq \gamma_3$, by Remark 3.2, Proposition 3.3, and assumptions (H3), (H4), the cost functional $J^\varepsilon(m^\varepsilon(\cdot))$ is well-defined and finite for any $\varepsilon > 0$ and $m^\varepsilon(\cdot) \in \mathcal{R}^\varepsilon$. In fact,

$$\begin{aligned}|J^\varepsilon(m^\varepsilon(\cdot))| \leq C \int_0^{T_0} &\mathbb{E}[1 + |X^\varepsilon(s)|^{\gamma_3} + |Y^\varepsilon(s)| + W_2(\mathscr{L}(X^\varepsilon(s)), \delta_0)\\ &+W_2(\mathscr{L}(Y^\varepsilon(s)), \delta_0)]ds + C\mathbb{E}[1 + |Y^\varepsilon(T_0)| + W_2(\mathscr{L}(Y^\varepsilon(T_0)), \delta_0)]\\ \leq C_{T_0}&.\end{aligned}$$

The proof of Proposition 3.3 is given in Appendix B.

With respect to the invariant probability measure μ in Proposition 3.1, define

$$\begin{aligned}&\bar{f}_2(y, \nu) = \int_{\mathbb{R}^{d_1}} f_2(x, \mu, y, \nu)\mu(dx), \quad \bar{G}(y, \nu) = \int_{\mathbb{R}^{d_1}} (gg^\top)(x, \mu, y, \nu)\mu(dx),\\ &\bar{R}_2(y, \nu) = \int_{\mathbb{R}^{d_1}} R_2(x, \mu, y, \nu)\mu(dx).\end{aligned}$$

Then, under $p \geq \max\{4\gamma_1, 4\gamma_2, \gamma_3\}$, by assumptions (H3), (H4) and Proposition 3.1, there exist positive constants $\bar{L}, \bar{K}$ such that

$$\begin{aligned}&|\bar{f}_2(y_1, \nu_1) - \bar{f}_2(y_2, \nu_2)| \vee |\bar{R}_2(y_1, \nu_1) - \bar{R}_2(y_2, \nu_2)| \leq \bar{L}(|y_1 - y_2| + W_2(\nu_1, \nu_2)),\\ &\|\bar{G}(y_1, \nu_1) - \bar{G}(y_2, \nu_2)\|\\ &\quad \leq \bar{L}(1 + |y_1| + |y_2| + W_2(\nu_1, \delta_0) + W_2(\nu_2, \delta_0))(|y_1 - y_2| + W_2(\nu_1, \nu_2)),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&|\bar{f}_2(y, \nu)| \vee |\bar{R}_2(y, \nu)| \leq \bar{K}(1 + |y| + W_2(\nu, \delta_0)),\\ &\|\bar{G}(y, \nu)\| \leq \bar{K}(1 + |y|^2 + W_2(\nu, \delta_0)^2),\end{aligned} \tag{3.8}$$

where $y, y_1, y_2 \in \mathbb{R}^{d_2}$ and $\nu, \nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2})$. Moreover, the controlled system in the limit problem can now be written as

$$\begin{aligned}dY(t) = \Big(\int_U f_1(Y(t), \mathscr{L}(Y(t)), u)m_t(du)\Big)dt &+ \bar{f}_2(Y(t), \mathscr{L}(Y(t)))dt\\ +\bar{g}(Y(t), \mathscr{L}(Y(t)))dW(t), &\qquad Y(0) = \zeta,\end{aligned} \tag{3.9}$$

and the cost functional is given by

$$\begin{aligned}J(m(\cdot)) = \mathbb{E}\Big[\int_0^{T_0} &\Big(\int_U R_1(Y(s), \mathscr{L}(Y(s)), u)m_s(du) + \bar{R}_2(Y(s), \mathscr{L}(Y(s)))\Big)ds\Big]\\ &+\mathbb{E}[Q(Y(T_0), \mathscr{L}(Y(T_0)))],\end{aligned} \tag{3.10}$$

where $\bar{g}$ satisfies $\bar{g} \cdot \bar{g}^\top = \bar{G}$ and $W(t)$ is a d_2-dimensional Brownian motion. In (3.9) and (3.10), $m(\cdot)$ is an admissible relaxed control associated with $W(\cdot)$, ξ, and ζ, i.e., admissible pair $(m(\cdot), W(\cdot), \xi, \zeta)$. For convenience of the notation, we shall omit the corresponding Brownian motion and initial values in what follows. Let $\mathcal{R}^0$ denote the collection of all admissible relaxed controls for the limit problem:

$$\begin{aligned}\mathcal{R}^0 = \{m(\cdot) : m(\cdot) \text{ is } &\mathcal{R}(U \times [0, \infty))\text{-valued random variable},\\ &\text{and } m(B, t) \text{ is } \{\mathcal{G}_t\}\text{-progressively measurable for any } B \in \mathcal{B}(U)\},\end{aligned}$$

where $\mathcal{G}_t = \sigma\{W(s), \xi, \zeta; s \le t\}$. Similar to (3.3), we can also define the optimal value related to the limit problem as

$$v^0 = \inf_{m(\cdot)\in\mathcal{R}^0} J(m(\cdot)). \tag{3.11}$$

For any admissible relaxed control $m(\cdot)$, by assumptions (H3), (H4), (3.7), (3.8), the coefficients f_1, $\bar{f}_2$, and $\bar{g}$ are continuous and linearly growing with respect to (y, ν). This together with the proof of Proposition 3.1 implies that there exists a weak solution (still denote it by $Y(\cdot)$) to (3.9). Moreover, for any $0 \le s, t \le T_0$, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T_0} |Y(t)|^4\Big] \le C_{T_0},\ \ \mathbb{E}|Y(t) - Y(s)|^4 \le C_{T_0}|t-s|^2,\ \text{ and } |J(m(\cdot))| \le C_{T_0}. \tag{3.12}$$

Therefore, the limit problem (3.9)–(3.11) is well-defined.

4. Weak convergence and near optimality.

In this section, we prove the near optimality of the original problem (1.1)–(1.3) by using the nearly optimal control of the limit problem (3.9)–(3.11). To this end, we first need to prove the weak convergence of the slow process and the relaxed control process in the revised problem with relaxed control (3.1)–(3.3). Then, we establish the nearly optimal control of the limit problem.

For any $u \in U$, $y \in \mathbb{R}^{d_2}$, $\nu \in \mathcal{P}_2(\mathbb{R}^{d_2})$, and $V \in C_c^\infty(\mathbb{R}^{d_2}; \mathbb{R})$, define an operator $\bar{L}^u$ by

$$\bar{L}^u(y,\nu)V(y) = \langle \nabla_y V(y), f_1(y,\nu,u) + \bar{f}_2(y,\nu)\rangle + \frac{1}{2}\operatorname{tr}[\nabla_y^2 V(y)\cdot \bar{G}(y,\nu)]. \tag{4.1}$$

A stochastic process $(Y(t), m(t))_{t\ge 0}$ is said to satisfy the martingale problem with operator $\bar{L}^u$ if, for any $V \in C_c^\infty(\mathbb{R}^{d_2}; \mathbb{R})$,

$$M_V(t) := V(Y(t)) - V(Y(0)) - \int_0^t \int_U \bar{L}^u(Y(s), \mathscr{L}(Y(s)))V(Y(s)) m_s(du)ds \tag{4.2}$$

is a martingale. Note that the existence of a weak solution to (3.9) is equivalent to the existence of a solution to the martingale problem (4.1)–(4.2). Then, in what follows, we shall use this martingale problem to prove that a given stochastic process satisfies (3.9).

4.1. Weak convergence.

In this subsection, we mainly prove that the sequence $\{(Y^\varepsilon(\cdot), m^\varepsilon(\cdot))\}_{0<\varepsilon\le 1}$ converges weakly to a stochastic process which is a weak solution to (3.9). This conclusion is collected in the following theorem.

THEOREM 4.1. *Let assumptions* (H1)–(H4) *hold,* $p \ge \max\{4\gamma_1, 4\gamma_2, \gamma_3\}$*, and* $\{m^\varepsilon(\cdot)\}_{0<\varepsilon\le 1}$ *be admissible relaxed control. Then, the sequence* $\{(Y^\varepsilon(\cdot), m^\varepsilon(\cdot))\}_{0<\varepsilon\le 1}$ *is tight in* $C([0,T_0];\mathbb{R}^{d_2}) \times \mathcal{R}(U \times [0,T_0])$*, and the limit of any weakly convergent subsequence (indexed by* ε_n*) satisfies* (3.9)*. Moreover, if* $m^{\varepsilon_n}(\cdot) \Rightarrow m(\cdot)$ *as* $n \to \infty$*, then* $m(\cdot) \in \mathcal{R}^0$ *and*

$$J^{\varepsilon_n}(m^{\varepsilon_n}(\cdot)) \to J(m(\cdot)) \ \text{ as } n \to \infty.$$

To prove Theorem 4.1, we need the following two lemmas. The first lemma illustrates the tightness criterion of the random variables valued in $C([0,T_0];\mathbb{R}^{d_2})$; see [25] for the proof.

LEMMA 4.2. *The sequence $\{Y^\varepsilon(\cdot)\}_{0<\varepsilon\le 1}$ is tight in $C([0,T_0];\mathbb{R}^{d_2})$ if and only if the following two conditions hold:*

(i) *there exists a constant $q>0$ such that*

$$\sup_{0<\varepsilon\le 1}\mathbb{E}|Y^\varepsilon(0)|^q<\infty; \tag{4.3}$$

(ii) *there exist constants $\alpha,\beta>0$ such that for any $0\le s,t\le T_0$,*

$$\sup_{0<\varepsilon\le 1}\mathbb{E}|Y^\varepsilon(t)-Y^\varepsilon(s)|^\alpha\le C_{T_0}|t-s|^{1+\beta}. \tag{4.4}$$

The lemma below states that the sequence of the random variables valued in $C([0,T_0];\mathbb{R}^{d_2})$, which is relatively compact, can be approximated by step functions; see [33] for the proof.

LEMMA 4.3. *Assume $Y^{\varepsilon_n}\Rightarrow Y$ in $C([0,T_0];\mathbb{R}^{d_2})$. Then, for any $\eta>0$, there are $k\ge 1$ and $\mathbb{R}^{d_2}$-valued step random functions y_1, y_2, ..., y_k such that*

$$\mathbb{P}\Big(\bigcap_{\ell=1}^k\Big\{\sup_{0\le t\le T_0}|Y^{\varepsilon_n}(t)-y_\ell(t)|>\eta\Big\}\Big)<\eta\quad\text{for any } n\ge 1,$$
$$\mathbb{P}\Big(\bigcap_{\ell=1}^k\Big\{\sup_{0\le t\le T_0}|Y(t)-y_\ell(t)|>\eta\Big\}\Big)<\eta.$$

We are now in position to prove Theorem 4.1.

Proof. We divide this proof into three steps.

Step 1*: Tightness.* Owing to the compactness of U, $U\times[0,T_0]$ is compact, complete, and separable. As a result, $\{m^\varepsilon(\cdot)\}_{0<\varepsilon\le 1}$ is tight in $\mathcal{R}(U\times[0,T_0])$. Recall that $Y^\varepsilon(0)=\zeta$ and $\mathbb{E}|\zeta|^4<\infty$. Then, (4.3) holds. Note that

$$\begin{aligned}Y^\varepsilon(t)=Y^\varepsilon(s)&+\int_s^t\int_U f(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon,u)m_r^\varepsilon(du)dr\\&+\int_s^t g(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon)dW^2(r),\end{aligned}$$

where $\mu_r^\varepsilon=\mathcal{L}(X^\varepsilon(r))$ and $\nu_r^\varepsilon=\mathcal{L}(Y^\varepsilon(r))$. Then, for any $s,t\in[0,T_0]$, by the Burkholder–Davis–Gundy inequality (see [25, Theorem 3.28, pp. 166] or [30, Theorem 1.7.3, p. 40]), assumption (H4), Remark 3.2, and Proposition 3.3, we have

$$\begin{aligned}&\mathbb{E}|Y^\varepsilon(t)-Y^\varepsilon(s)|^4\\&=\mathbb{E}\Big|\int_s^t\int_U f(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon,u)m_r^\varepsilon(du)dr+\int_s^t g(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon)dW^2(r)\Big|^4\\&\le C|t-s|^3\cdot\mathbb{E}\Big[\int_s^t\int_U|f(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon,u)|^4m_r^\varepsilon(du)dr\Big]\\&\quad+C|t-s|\cdot\mathbb{E}\Big[\int_s^t\|g(X^\varepsilon(r),\mu_r^\varepsilon,Y^\varepsilon(r),\nu_r^\varepsilon)\|^4dr\Big]\\&\le C_{T_0}|t-s|\cdot\mathbb{E}\Big[\int_s^t(1+|X^\varepsilon(r)|^{4\gamma}+|Y^\varepsilon(r)|^4+W_2(\mu_r^\varepsilon,\delta_0)^4+W_2(\nu_r^\varepsilon,\delta_0)^4)dr\Big]\\&\le C_{T_0}|t-s|^2,\end{aligned}$$

where $\gamma = \max\{\gamma_1, \gamma_2\}$ and C_{T_0} is independent of ε. Therefore, (4.4) holds, and the tightness of $\{Y^\varepsilon(\cdot)\}_{0<\varepsilon\le 1}$ follows.

Step 2: Limit point satisfies the martingale problem (4.1)–(4.2). For any fixed $T_0 > 0$, by the Prokhorov theorem (see [22, Theorem 2, p. 28]), we can extract a weakly convergent subsequence and use ε_n as its index. Let $(Y(\cdot), m(\cdot))$ be the corresponding limit process. Then, by the Skorohod representation theorem (see [22, Theorem 3, p. 29]), without changing notation, we get that $\mathbb{P}$-a.s.,

$$\sup_{0\le t\le T_0} |Y^{\varepsilon_n}(t) - Y(t)| \to 0, \quad m^{\varepsilon_n}(\cdot) \Rightarrow m(\cdot) \quad \text{as } n\to\infty.$$

In addition, by Proposition 3.3 and the Vitali convergence theorem (see [1, Theorem 4.5.4, p. 268] or [9, Theorem 4.5.4, p. 101]), we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T_0} |Y^{\varepsilon_n}(t) - Y(t)|^3\Big] \to 0 \quad \text{as } n\to\infty \tag{4.5}$$

and then

$$\mathbb{E}\Big[\sup_{0\le t\le T_0} |Y(t)|^3\Big] \le C_{T_0}. \tag{4.6}$$

Applying the definition of martingale, it is sufficient to prove that for any $0 \le s < t \le T_0$, integer $k_1, k_2 \ge 1$, $0 \le s_1 \le \cdots \le s_{k_1} \le s$, $\varphi_1(\cdot,\cdot), \ldots, \varphi_{k_2}(\cdot,\cdot) \in C_b(U\times[0,T_0])$, bounded Lipschitz continuous function $h: \mathbb{R}^{k_1 d_2 + k_1 k_2 + d_1} \to \mathbb{R}$, and $V \in C_c^\infty(\mathbb{R}^{d_2};\mathbb{R})$, the following holds:

$$\begin{aligned}
\mathbb{E}[M_V(s,t,h)] &:= \mathbb{E}\Big[\Big(V(Y(t)) - V(Y(s)) - \int_s^t \int_U \bar{L}^u(Y(r),\nu_r)V(Y(r))m_r(du)dr\Big) \\
&\qquad \times h(Y(s_i), (m,\varphi_j)_{s_i}, \xi; i\le k_1, j\le k_2)\Big] = 0,
\end{aligned} \tag{4.7}$$

where $\nu_r = \mathscr{L}(Y(r))$. To proceed, we apply the Itô formula to the function $V(\cdot)$ and obtain

$$\begin{aligned}
&\mathbb{E}[M_V^n(s,t,h)] \\
&:= \mathbb{E}\Big[\Big(V(Y^{\varepsilon_n}(t)) - V(Y^{\varepsilon_n}(s)) - \int_s^t \int_U \langle \nabla_y V(Y^{\varepsilon_n}(r)), f_1(Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n}, u)\rangle m_r^{\varepsilon_n}(du)dr \\
&\qquad - \int_s^t \langle \nabla_y V(Y^{\varepsilon_n}(r)), f_2(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n})\rangle dr \\
&\qquad - \frac{1}{2}\int_s^t \mathrm{tr}[\nabla_y^2 V(Y^{\varepsilon_n}(r)) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n})]dr\Big) \\
&\qquad \times h(Y^{\varepsilon_n}(s_i), (m^{\varepsilon_n},\varphi_j)_{s_i}, \xi; i\le k_1, j\le k_2)\Big] \\
&= \mathbb{E}\Big[\int_s^t \langle \nabla_y V(Y^{\varepsilon_n}(r)), g(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n}) dW^2(r)\rangle \\
&\qquad \times h(Y^{\varepsilon_n}(s_i), (m^{\varepsilon_n},\varphi_j)_{s_i}, \xi; i\le k_1, j\le k_2)\Big] = 0,
\end{aligned}$$

where we have used the fact that

$$\Big\{\int_0^t \langle \nabla_y V(Y^{\varepsilon_n}(r)), g(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n}) dW^2(r)\rangle, \mathcal{F}_t; t\ge 0\Big\}$$

is a martingale. This together with (4.7) implies that we need only prove

$$\mathbb{E}[M_V^n(s,t,h)] \to \mathbb{E}[M_V(s,t,h)] \quad \text{as } n \to \infty.$$

Note that $m^{\varepsilon_n}(\cdot) \Rightarrow m(\cdot)$ implies $(m^{\varepsilon_n}, \varphi_j)_{s_i} \to (m, \varphi_j)_{s_i}$ for all $i \le k_1, j \le k_2$. Then, by the Lebesgue dominated convergence theorem,

$$\mathbb{E}|(m^{\varepsilon_n}, \varphi_j)_{s_i} - (m, \varphi_j)_{s_i}|^2 \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

Furthermore, by (4.5), (4.8), and the properties of h and V, one can obtain

$$\begin{aligned} &|\mathbb{E}[(V(Y^{\varepsilon_n}(t)) - V(Y^{\varepsilon_n}(s)))h(Y^{\varepsilon_n}(s_i), (m^{\varepsilon_n}, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)] \\ &\quad -\mathbb{E}[(V(Y(t)) - V(Y(s)))h(Y(s_i), (m, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)]| \\ &\quad \le C\mathbb{E}\Big[\sup_{0\le t\le T_0} |Y^{\varepsilon_n}(t) - Y(t)|\Big] + \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbb{E}|(m^{\varepsilon_n}, \varphi_j)_{s_i} - (m, \varphi_j)_{s_i}| \\ &\quad \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{4.9}$$

Moreover, by assumptions (H3), (H4), (4.5), (4.6), (4.8), and the convergence of $m^{\varepsilon_n}(\cdot)$, we get

$$\begin{aligned} &\Big|\mathbb{E}\Big[\Big(\int_s^t \int_U \langle \nabla_y V(Y^{\varepsilon_n}(r)), f_1(Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n}, u)\rangle m_r^{\varepsilon_n}(du)dr\Big) \\ &\qquad \times h(Y^{\varepsilon_n}(s_i), (m^{\varepsilon_n}, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)\Big] \\ &\quad -\mathbb{E}\Big[\Big(\int_s^t \int_U \langle \nabla_y V(Y(r)), f_1(Y(r), \nu_r, u)\rangle m_r(du)dr\Big) \\ &\qquad \times h(Y(s_i), (m, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)]\Big| \\ &\le \mathbb{E}\Big|\int_s^t \int_U \langle \nabla_y V(Y(r)), f_1(Y(r), \nu_r, u)\rangle (m^{\varepsilon_n}(dudr) - m(dudr))\Big| \\ &\qquad + C_{T_0}\Big(\mathbb{E}\Big[\sup_{0\le t\le T_0} |Y^{\varepsilon_n}(t) - Y(t)|^2\Big]\Big)^{\frac{1}{2}} + \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbb{E}|(m^{\varepsilon_n}, \varphi_j)_{s_i} - (m, \varphi_j)_{s_i}| \\ &\to 0 \qquad \text{as } n \to \infty, \end{aligned} \tag{4.10}$$

where we have used the fact that for almost all $\omega \in \Omega$, $\langle \nabla_y V(Y(r)), f_1(Y(r), \nu_r, u)\rangle$ is a bounded and continuous function in (r, u). Therefore, according to (4.9) and (4.10), it remains to prove

$$\begin{aligned} &\mathbb{E}[Z_V^n(s,t,h)] := \mathbb{E}\Big[\Big(\int_s^t \langle \nabla_y V(Y^{\varepsilon_n}(r)), f_2(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n})\rangle dr \\ &+\frac{1}{2}\int_s^t \mathrm{tr}[\nabla_y^2 V(Y^{\varepsilon_n}(r)) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, Y^{\varepsilon_n}(r), \nu_r^{\varepsilon_n})]dr\Big) \\ &\qquad \times h(Y^{\varepsilon_n}(s_i), (m^{\varepsilon_n}, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)\Big] \\ &\to \mathbb{E}\Big[\Big(\int_s^t \langle \nabla_y V(Y(r)), \bar{f}_2(Y(r), \nu_r)\rangle dr + \frac{1}{2}\int_s^t \mathrm{tr}[\nabla_y^2 V(Y(r)) \cdot \bar{G}(Y(r), \nu_r)]dr\Big) \\ &\quad \times h(Y(s_i), (m, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2)\Big] =: \mathbb{E}[Z_V(s,t,h)] \quad \text{as } n \to \infty. \end{aligned} \tag{4.11}$$

For any $\eta > 0$, by using step functions $y_1, \ldots, y_k$ in Lemma 4.3, we can choose continuous functions $\Phi, \Phi_1, \ldots, \Phi_k : C([0,T_0];\mathbb{R}^{d_2}) \to [0,1]$ such that

(i) $\Phi(y) + \Sigma_{i=\ell}^k \Phi_\ell(y) = 1$ for any $y \in C([0,T_0];\mathbb{R}^{d_2})$;

(ii) $\mathrm{supp}\,(\Phi) \subseteq \bigcap_{\ell=1}^k \{y \in C([0,T_0];\mathbb{R}^{d_2}) : \kappa_\ell(y) > \eta\}$;

(iii) $\mathrm{supp}\,(\Phi_\ell) \subseteq \{y \in C([0,T_0];\mathbb{R}^{d_2}) : \kappa_\ell(y) < 2\eta\}$ for all $1 \le \ell \le k$,

where $\mathrm{supp}\,(\Phi)$ denotes the support set of the function Φ and $\kappa_\ell(y) = \sup_{0\le t\le T_0} |y(t) - y_\ell(t)|$ for every $y \in C([0,T_0];\mathbb{R}^{d_2})$; see [33] for more details. Let

$$\varrho_n = \Phi(Y^{\varepsilon_n}), \quad \varrho = \Phi(Y), \quad \zeta_{\ell,n} = \Phi_\ell(Y^{\varepsilon_n}), \quad \zeta_\ell = \Phi_\ell(Y).$$

Then, by virtue of Lemma 4.3 again, one has that for any $n \ge 1$,

$$\begin{aligned}
&\mathbb{P}(\mathrm{supp}\,(\varrho_n)) \le \mathbb{P}\Big(\bigcap_{\ell=1}^k \Big\{\omega \in \Omega : \sup_{0\le t\le T_0} |Y^{\varepsilon_n}(t) - y_\ell(t)| > \eta\Big\}\Big) < \eta,\\
&\mathbb{P}(\mathrm{supp}\,(\varrho)) \le \mathbb{P}\Big(\bigcap_{\ell=1}^k \Big\{\omega \in \Omega : \sup_{0\le t\le T_0} |Y(t) - y_\ell(t)| > \eta\Big\}\Big) < \eta.
\end{aligned} \tag{4.12}$$

Moreover, applying $\varrho_n + \Sigma_{i=\ell}^k \zeta_{\ell,n} = 1$ and $\varrho + \Sigma_{i=\ell}^k \zeta_\ell = 1$, we get

$$\begin{aligned}
|\mathbb{E}[Z_V^n(s,t,h)] - \mathbb{E}[Z_V(s,t,h)]| &\le \mathbb{E}[|Z_V^n(s,t,h)|\varrho_n] + \mathbb{E}[|Z_V(s,t,h)|\varrho]\\
&\quad + \Big|\sum_{\ell=1}^k \mathbb{E}[Z_V^n(s,t,h)\zeta_{\ell,n}] - \sum_{\ell=1}^k \mathbb{E}[Z_V(s,t,h)\zeta_\ell]\Big|.
\end{aligned} \tag{4.13}$$

Note that $\varrho_n \le 1$. Then, by assumption (H4), Remark 3.2, Proposition 3.3, (4.12), the Hölder inequality, and the properties of h and V, one can get

$$\begin{aligned}
&\mathbb{E}[|Z_V^n(s,t,h)|\varrho_n]\\
&\le (\mathbb{E}|Z_V^n(s,t,h)|^2)^{\frac12}(\mathbb{E}\varrho_n^2)^{\frac12}\\
&\le C_{T_0}\Big(1 + \int_s^t \big(\mathbb{E}|X^{\varepsilon_n}(r)|^{4\gamma} + (\mathbb{E}|X^{\varepsilon_n}(r)|^2)^2 + (\mathbb{E}|Y^{\varepsilon_n}(r)|^2)^2\big)dr\Big)^{\frac12}\\
&\quad \times \big(\mathbb{P}(\mathrm{supp}\,(\varrho_n))\big)^{\frac12}\\
&\le C_{T_0}\Big(1 + \int_s^t \mathbb{E}|X^{\varepsilon_n}(r)|^{4\gamma} dr\Big)^{\frac12} \cdot \eta^{\frac12} \le C_{T_0}\eta^{\frac12},
\end{aligned} \tag{4.14}$$

According to (3.8), (4.6), (4.12), and using the same technique as (4.14), we have

$$\mathbb{E}[|Z_V(s,t,h)|\varrho] \le (\mathbb{E}|Z_V(s,t,h)|^2)^{\frac12}(\mathbb{E}\varrho^2)^{\frac12} \le C_{T_0}\eta^{\frac12}. \tag{4.15}$$

Let

$$\begin{aligned}
Z_V^{\ell,n}(s,t,h) &= \Big(\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, y_\ell(r), \nu_r^{\varepsilon_n})\rangle dr + \frac12 \int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r))\\
&\quad \times (gg^\top)(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, y_\ell(r), \nu_r^{\varepsilon_n})]dr\Big) h(y_\ell(s_i), (m^{\varepsilon_n}, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2),\\
Z_V^\ell(s,t,h) &= \Big(\int_s^t \langle \nabla_y V(y_\ell(r)), \bar f_2(y_\ell(r), \nu_r)\rangle dr + \frac12 \int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot \bar G(y_\ell(r), \nu_r)]dr\Big)\\
&\quad \times h(y_\ell(s_i), (m, \varphi_j)_{s_i}, \xi; i \le k_1, j \le k_2).
\end{aligned}$$

Then, we have

$$
\begin{aligned}
&\Big|\sum_{\ell=1}^{k}\mathbb{E}[Z_V^n(s,t,h)\zeta_{\ell,n}]-\sum_{\ell=1}^{k}\mathbb{E}[Z_V(s,t,h)\zeta_\ell]\Big|\\
&\le\sum_{\ell=1}^{k}\mathbb{E}[|Z_V^n(s,t,h)-Z_V^{\ell,n}(s,t,h)|\zeta_{\ell,n}]+\sum_{\ell=1}^{k}\mathbb{E}[|Z_V(s,t,h)-Z_V^{\ell}(s,t,h)|\zeta_\ell]\\
&\quad+\sum_{\ell=1}^{k}|\mathbb{E}[Z_V^{\ell,n}(s,t,h)\zeta_{\ell,n}]-\mathbb{E}[Z^{\ell}(s,t,h)\zeta_\ell]|\\
&=: I_1+I_2+I_3. \qquad (4.16)
\end{aligned}
$$

Let us estimate I_1, I_2, and I_3 one by one. According to assumptions (H3), (H4), Remark 3.2, Proposition 3.3, and the properties of h and V, we get

$$
\begin{aligned}
&|Z_V^n(s,t,h)-Z_V^{\ell,n}(s,t,h)|\\
&\le\Big|\int_s^t\langle\nabla_yV(Y^{\varepsilon_n}(r)),f_2(X^{\varepsilon_n}(r),\mu_r^{\varepsilon_n},Y^{\varepsilon_n}(r),\nu_r^{\varepsilon_n})\rangle dr\cdot h(Y^{\varepsilon_n}(s_i),(m^{\varepsilon_n},\varphi_j)_{s_i},\xi;i,j)\\
&\quad-\int_s^t\langle\nabla_yV(y_\ell(r)),f_2(X^{\varepsilon_n}(r),\mu_r^{\varepsilon_n},y_\ell(r),\nu_r^{\varepsilon_n})\rangle dr\cdot h(y_\ell(s_i),(m^{\varepsilon_n},\varphi_j)_{s_i},\xi;i,j)\Big|\\
&\quad+\frac{1}{2}\Big|\int_s^t\mathrm{tr}[\nabla_y^2V(Y^{\varepsilon_n}(r))\cdot(gg^\top)(X^{\varepsilon_n}(r),\mu_r^{\varepsilon_n},Y^{\varepsilon_n}(r),\nu_r^{\varepsilon_n})]dr\\
&\qquad\times h(Y^{\varepsilon_n}(s_i),(m^{\varepsilon_n},\varphi_j)_{s_i},\xi;i,j)\\
&\quad-\int_s^t\mathrm{tr}[\nabla_y^2V(y_\ell(r))\cdot(gg^\top)(X^{\varepsilon_n}(r),\mu_r^{\varepsilon_n},y_\ell(r),\nu_r^{\varepsilon_n})]dr\cdot h(y_\ell(s_i),(m^{\varepsilon_n},\varphi_j)_{s_i},\xi;i,j)\Big|\\
&\le C_{T_0}\int_s^t(1+|X^{\varepsilon_n}(r)|^{2\gamma}+|Y^{\varepsilon_n}(r)|^2)dr\cdot\sup_{0\le t\le T_0}|Y^{\varepsilon_n}(t)-y_\ell(t)|.
\end{aligned}
$$

Note that $\sum_{\ell=1}^k\zeta_{\ell,n}\le 1$ and $\mathrm{supp}\,(\zeta_{\ell,n})\subseteq\{\omega\in\Omega:\sup_{0\le t\le T_0}|Y^{\varepsilon_n}(t)-y_\ell(t)|<2\eta\}$. Then, by Remark 3.2 and Proposition 3.3, one can calculate

$$
\begin{aligned}
I_1&\le C_{T_0}\sum_{\ell=1}^{k}\int_s^t\mathbb{E}\Big[(1+|X^{\varepsilon_n}(r)|^{2\gamma}+|Y^{\varepsilon_n}(r)|^2)\Big(\sup_{0\le t\le T_0}|Y^{\varepsilon_n}(t)-y_\ell(t)|\Big)\zeta_{\ell,n}\Big]dr\\
&\le C_{T_0}\eta. \qquad (4.17)
\end{aligned}
$$

Recall that $\sum_{\ell=1}^k\zeta_\ell\le 1$ and $\mathrm{supp}\,(\zeta_\ell)\subseteq\{\omega\in\Omega:\sup_{0\le t\le T_0}|Y(t)-y_\ell(t)|<2\eta\}$. Then, by (3.7), (3.8), (4.6), and using a similar technique as (4.17), we have

$$
I_2\le C_{T_0}\sum_{\ell=1}^{k}\int_s^t\mathbb{E}\Big[(1+|Y(r)|^2)\Big(\sup_{0\le t\le T_0}|Y(t)-y_\ell(t)|\Big)\zeta_\ell\Big]dr\le C_{T_0}\eta. \qquad (4.18)
$$

In the following, we estimate I_3. Note that $\zeta_{\ell,n}\le 1$ and $Z_V^\ell(s,t,h)$ is bounded. Then, for any $1\le\ell\le k$,

$$
\begin{aligned}
&|\mathbb{E}[Z_V^{\ell,n}(s,t,h)\zeta_{\ell,n}]-\mathbb{E}[Z_V^{\ell}(s,t,h)\zeta_\ell]|\\
&\le\mathbb{E}|Z_V^{\ell,n}(s,t,h)-Z_V^{\ell}(s,t,h)|+C_{T_0}\mathbb{E}|\zeta_{\ell,n}-\zeta_\ell|=:I_{31}^\ell+I_{32}^\ell. \qquad (4.19)
\end{aligned}
$$

For any $1 \leq \ell \leq k$, according to $\zeta_{\ell,n} = \Phi_\ell(Y^{\varepsilon_n})$, $\zeta_\ell = \Phi_\ell(Y)$, and the fact that Φ_ℓ is a bounded continuous function, one can get that $\mathbb{P}$-a.s., $\zeta_{\ell,n} \to \zeta_\ell$. Consequently, by the Lebesgue dominated convergence theorem, $\zeta_{\ell,n} \leq 1$ implies that

$$I_{32}^\ell \to 0 \quad \text{as } n \to \infty. \tag{4.20}$$

Therefore, it only remains to consider the term I_{31}^ℓ. To proceed, we introduce the following random variable as a bridge:

$$\begin{aligned}\hat{Z}_V^{\ell,n}(s,t,h) = \Big(\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r)\rangle dr + \frac{1}{2}\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \\ \times (gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r)]dr\Big) h(y_\ell(s_i), (m, \varphi_j)_{s_i}, \xi; i \leq k_1, j \leq k_2),\end{aligned}$$

where μ is the invariant probability measure in Proposition 3.1. Then,

$$I_{31}^\ell \leq \mathbb{E}|Z_V^{\ell,n}(s,t,h) - \hat{Z}_V^{\ell,n}(s,t,h)| + \mathbb{E}|\hat{Z}_V^{\ell,n}(s,t,h) - Z_V^\ell(s,t,h)|.$$

On the one hand, applying assumptions (H3), (H4), Proposition 3.1, Remark 3.2, Proposition 3.3, (4.5), (4.6), (4.8), and the properties of h and V, one has

$$\begin{aligned}&\mathbb{E}|Z_V^{\ell,n}(s,t,h) - \hat{Z}_V^{\ell,n}(s,t,h)| \\ &\quad\leq C\int_s^t \mathbb{E}[|\nabla_y V(y_\ell(r))| \cdot |f_2(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, y_\ell(r), \nu_r^{\varepsilon_n}) \\ &\qquad\qquad -f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r)|]dr \\ &\quad\quad +C\int_s^t \mathbb{E}[\|\nabla_y^2 V(y_\ell(r))\| \cdot \|(gg^\top)(X^{\varepsilon_n}(r), \mu_r^{\varepsilon_n}, y_\ell(r), \nu_r^{\varepsilon_n}) \\ &\qquad\qquad -(gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r)\|]dr \\ &\quad\quad +C_{T_0}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(\mathbb{E}|(m^{\varepsilon_n}, \varphi_j)_{s_i} - (m, \varphi_j)_{s_i}|^2)^{\frac{1}{2}} \\ &\quad\leq C_{T_0}\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T_0}|Y^{\varepsilon_n}(t) - Y(t)|^2\Big]\Big)^{\frac{1}{2}} + C_{T_0}\int_s^t W_2(P^*_{\frac{r}{\varepsilon_n}}\mu_\xi, \mu)dr \\ &\quad\quad +C_{T_0}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(\mathbb{E}|(m^{\varepsilon_n}, \varphi_j)_{s_i} - (m, \varphi_j)_{s_i}|^2)^{\frac{1}{2}} \\ &\quad\to 0 \quad \text{as } n \to \infty,\end{aligned} \tag{4.21}$$

in which $\mu_\xi = \mathscr{L}(\xi)$. On the other hand, we consider the term $\mathbb{E}|\hat{Z}_V^{\ell,n}(s,t,h) - Z_V^\ell(s,t,h)|$. For a sufficiently large $N \geq 1$ (N is to be chosen later), we consider a partition $\{t_j^N\}_{0\leq j\leq N}$ of the time interval $[s,t]$. Let $\Delta_N = \max_{0\leq j\leq N-1}(t_{j+1}^N - t_j^N)$ be decreasing and converge to 0. Define $\lfloor s\rfloor_N = t_j^N$ for all $s \in [t_j^N, t_{j+1}^N)$. Then, we obtain

$$\begin{aligned}&\mathbb{E}|\hat{Z}_V^{\ell,n}(s,t,h) - Z_V^\ell(s,t,h)| \\ &\quad\leq C\mathbb{E}\Big|\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r) - f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r\rfloor_N})\rangle dr\Big| \\ &\quad\quad + C\mathbb{E}\Big|\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r\rfloor_N}) - \bar{f}_2(y_\ell(r), \nu_{\lfloor r\rfloor_N})\rangle dr\Big|\end{aligned}$$

$$
\begin{aligned}
&+ C\mathbb{E}\Big|\int_s^t \langle \nabla_y V(y_\ell(r)), \bar{f}_2(y_\ell(r), \nu_{\lfloor r \rfloor_N}) - \bar{f}_2(y_\ell(r), \nu_r)\rangle dr\Big| \\
&+ C\mathbb{E}\Big|\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot [(gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_r) \\
&\quad - (gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})]]dr\Big| \\
&+ C\mathbb{E}\Big|\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot [(gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N}) - \bar{G}(y_\ell(r), \nu_{\lfloor r \rfloor_N})]]dr\Big| \\
&+ C\mathbb{E}\Big|\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot (\bar{G}(y_\ell(r), \nu_{\lfloor r \rfloor_N}) - \bar{G}(y_\ell(r), \nu_r))]dr\Big| \\
&=: J_{11}^\ell + J_{12}^\ell + J_{13}^\ell + J_{21}^\ell + J_{22}^\ell + J_{23}^\ell,
\end{aligned}
$$

where $\nu_{\lfloor r \rfloor_N} = \mathscr{L}(Y(\lfloor r \rfloor_N))$. In the following, we estimate these terms one by one. According to assumption (H3) and (3.7), one can get

$$
J_{11}^\ell + J_{13}^\ell \le C\int_s^t W_2(\nu_{\lfloor r \rfloor_N}, \nu_r)dr \le C_{T_0}\Big(\mathbb{E}\Big[\sup_{s\le r\le t}|Y(\lfloor r \rfloor_N) - Y(r)|^2\Big]\Big)^{\frac{1}{2}}.
$$

Note that $\mathbb{P}$-a.s.,

$$
\sup_{s\le r\le t}|Y(\lfloor r \rfloor_N) - Y(r)| \le \max_{0\le j\le N-1}\ \sup_{t_j^N \le r\le t_{j+1}^N}|Y(r) - Y(t_j^N)| \to 0 \ \text{ as } N\to\infty.
$$

Then, by (4.6) and the Vitali convergence theorem, we have

$$
J_{11}^\ell + J_{13}^\ell \to 0 \ \text{ as } N\to\infty. \tag{4.22}
$$

Applying assumptions (H3), (H4), (3.7), (4.6), Proposition 3.1, Remark 3.2 and using the same technique as $J_{11}^\ell + J_{13}^\ell$ yield that

$$
\begin{aligned}
J_{21}^\ell + J_{23}^\ell &\le C\int_s^t \mathbb{E}[\|\nabla_y^2 V(y_\ell(r))\| \cdot (1 + |X^{\varepsilon_n}(r)|^{\gamma_2} + |y_\ell(r)| + W_2(\mu, \delta_0) \\
&\qquad + W_2(\nu_{\lfloor r \rfloor_N}, \delta_0) + W_2(\nu_r, \delta_0)) \cdot W_2(\nu_{\lfloor r \rfloor_N}, \nu_r)]dr \\
&\le C_{T_0}\Big(\mathbb{E}\Big[\sup_{s\le r\le t}|Y(\lfloor r \rfloor_N) - Y(r)|^2\Big]\Big)^{\frac{1}{2}} \to 0 \ \text{ as } N\to\infty.
\end{aligned}
$$

This together with (4.22) implies that for any $\epsilon > 0$, there exists a sufficiently large N such that

$$
J_{11}^\ell + J_{13}^\ell + J_{21}^\ell + J_{23}^\ell \le \frac{\epsilon}{2}. \tag{4.23}
$$

Below, we estimate the terms J_{12}^ℓ and J_{22}^ℓ by using the ergodic theorem (see [12, Theorem 3.3.1, p. 30]). Since y_ℓ is a step function on $[0, T_0]$, we can reselect N and a partition of $[s, t]$, such that (4.23) holds, and on the interval (t_j^N, t_{j+1}^N), $y_\ell(r) = z_{\ell j}$ and $Y(\lfloor r \rfloor_N) = Y(t_j^N)$. Consequently,

$$
\begin{aligned}
&\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})\rangle dr \\
&= \sum_{j=0}^{N-1}\int_{t_j^N}^{t_{j+1}^N} \langle \nabla_y V(z_{\ell j}), f_2(X^{\varepsilon_n}(r), \mu, z_{\ell j}, \nu_{t_j^N})\rangle dr
\end{aligned}
$$

and

$$\begin{aligned}
&\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})]dr \\
&\quad = \sum_{j=0}^{N-1} \int_{t_j^N}^{t_{j+1}^N} \mathrm{tr}[\nabla_y^2 V(z_{\ell j}) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu, z_{\ell j}, \nu_{t_j^N})]dr.
\end{aligned}$$

By assumption (H4), Proposition 3.1, and the ergodic theorem, we have that $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\frac{1}{t_{j+1}^N - t_j^N} \int_{t_j^N}^{t_{j+1}^N} \langle \nabla_y V(z_{\ell j}), f_2(X^{\varepsilon_n}(r), \mu, z_{\ell j}, \nu_{t_j^N})\rangle dr \\
&\quad = \frac{\varepsilon_n}{t_{j+1}^N - t_j^N} \int_{\frac{t_j^N}{\varepsilon_n}}^{\frac{t_{j+1}^N}{\varepsilon_n}} \langle \nabla_y V(z_{\ell j}), f_2(X(r), \mu, z_{\ell j}, \nu_{t_j^N})\rangle dr \to \langle \nabla_y V(z_{\ell j}), \bar{f}_2(z_{\ell j}, \nu_{t_j^N})\rangle
\end{aligned}$$

and

$$\frac{1}{t_{j+1}^N - t_j^N} \int_{t_j^N}^{t_{j+1}^N} \mathrm{tr}[\nabla_y^2 V(z_{\ell j}) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu, z_{\ell j}, \nu_{t_j^N})]dr \to \mathrm{tr}[\nabla_y^2 V(z_{\ell j}) \bar{G}(z_{\ell j}, \nu_{t_j^N})].$$

Therefore, one can obtain that $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})\rangle dr \\
&\quad \to \int_s^t \langle \nabla_y V(y_\ell(r)), \bar{f}(y_i(r), \nu_{\lfloor r \rfloor_N})\rangle dr \quad \text{as } n \to \infty
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
&\int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})]dr \\
&\quad \to \int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot \bar{G}(y_\ell(r), \nu_{\lfloor r \rfloor_N})]dr \quad \text{as } n \to \infty.
\end{aligned} \tag{4.25}$$

Using assumption (H4), (4.6), Proposition 3.1, and Remark 3.2, we compute

$$\begin{aligned}
&\mathbb{E}\Big| \int_s^t \langle \nabla_y V(y_\ell(r)), f_2(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})\rangle dr \Big|^2 \\
&\le C \int_s^t \mathbb{E}(1 + |X^{\varepsilon_n}(r)|^{2\gamma_1} + W_2(\mu, \delta_0)^2 + W_2(\nu_{\lfloor r \rfloor_N}, \delta_0)^2) dr < \infty, \\
&\mathbb{E}\Big| \int_s^t \mathrm{tr}[\nabla_y^2 V(y_\ell(r)) \cdot (gg^\top)(X^{\varepsilon_n}(r), \mu, y_\ell(r), \nu_{\lfloor r \rfloor_N})]dr \Big|^2 < \infty.
\end{aligned}$$

This together with (4.23)–(4.25) implies that for any $\epsilon > 0$, there exists a sufficiently large n_0 such that for all $n \ge n_0$, $\mathbb{E}|\hat{Z}_V^{\ell,n}(s,t,h) - Z_V^\ell(s,t,h)| \le \epsilon$. By (4.19)–(4.21), one gets $I_3 \to 0$ as $n \to \infty$. Thus, it follows from (4.13)–(4.18) that for any $\eta > 0$,

$$\lim_{n\to\infty} |\mathbb{E}[Z_V^n(s,t,h)] - \mathbb{E}[Z_V(s,t,h)]| \le C_{T_0}\eta^{\frac{1}{2}} + C_{T_0}\eta + \lim_{n\to\infty} I_3 = C_{T_0}\eta^{\frac{1}{2}} + C_{T_0}\eta.$$

The arbitrariness of η implies that (4.11) follows. Therefore,

$$M_V(t) := V(Y(t)) - V(Y(0)) - \int_0^t \int_U \bar{L}^u(Y(s), \mathcal{L}(Y(s)))V(Y(s))m_s(du)ds$$

is a martingale with respect to $\mathcal{G}_t^0 := \sigma\{Y(s), m_s(\cdot), \xi; 0 \leq s \leq t\}$. Furthermore, $m(\cdot)$ is an admissible relaxed control for the limit problem (3.9)–(3.11). In fact, since $M_V(t)$ is a martingale with respect to $\mathcal{G}_t^0$, there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{G}}_t^0\}_{0\leq t\leq T_0}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \{\mathcal{G}_t^0\}_{0\leq t\leq T_0}, \mathbb{P})$ on which there lives an $\{\tilde{\mathcal{G}}_t^0\}$-adapted Brownian motion $\tilde{W}(\cdot)$, such that $(\tilde{Y}(\cdot), \tilde{m}(\cdot), \tilde{W}(\cdot))$ satisfies (3.9) and $\mathcal{L}(\tilde{Y}(0)) = \mathcal{L}(\zeta)$. Here,

$$\tilde{Y}(\tilde{\omega}) = Y(\theta\tilde{\omega}), \quad \tilde{m}(\tilde{\omega}) = m(\theta\tilde{\omega}),$$

and the mapping $\theta : (\tilde{\Omega}, \tilde{\mathcal{F}}) \to (\Omega, \mathcal{F})$; see [45, Definition 5.9, p. 40] for more details. Note that by the definition of extension of probability space, $\tilde{m}(t)$ and $\tilde{Y}(0)$ is measurable with respect to $\tilde{\mathcal{G}}_t^0$. Then, we consider the limit problem (3.9)–(3.11) on this new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{G}}_t^0\}_{0\leq t\leq T_0}, \tilde{\mathbb{P}})$. Moreover, we deduce that $\tilde{m}(\cdot)$ is admissible for the limit problem since $\tilde{W}(\cdot)$ is a $\{\tilde{\mathcal{G}}_t^0\}$-adapted Brownian motion and the initial conditions $\tilde{\xi}, \tilde{Y}(0)$ are measurable with respect to $\tilde{\mathcal{G}}_t^0$. Therefore, for convenience of notation, in this theorem, we use directly the original probability space and state that $m(\cdot)$ is admissible for the limit problem.

Step 3*: Convergence of the cost functional* $J^{\varepsilon_n}(\cdot)$*.* According to assumptions (H3), (H4), (3.8), and (4.6), $|J(m(\cdot))| \leq C_{T_0}$. Recall that

$$\begin{aligned} &J^{\varepsilon_n}(m^{\varepsilon_n}(\cdot)) \\ &= \mathbb{E}\Big[\int_0^{T_0}\Big(\int_U R_1(Y^{\varepsilon_n}(s), \nu_s^{\varepsilon_n}, u)m_s^{\varepsilon_n}(du) + R_2(X^{\varepsilon_n}(s), \mu_s^{\varepsilon_n}, Y^{\varepsilon_n}(s), \nu_s^{\varepsilon_n})\Big)ds\Big] \\ &\quad + \mathbb{E}[Q(Y^{\varepsilon_n}(T_0), \nu_{T_0}^{\varepsilon_n}))]. \end{aligned}$$

Then, by the same technique as Step 2, $J^{\varepsilon_n}(m^{\varepsilon_n}(\cdot)) \to J(m(\cdot))$ as $n \to \infty$ can be proved. So, we omit the proof. ∎

Remark 4.4. We can also prove the following result. Let $u(\cdot)$ be a feedback control with the Lipschitz continuous function $u_0(y, \nu)$, and let f_1, R_1 be Lipschitz continuous with respect to the control component u. Then, the original problem (1.1)–(1.3) and the limit problem (3.9)–(3.11) are well-defined under u_0, and $J^\varepsilon(u_0) \to J(u_0)$ as $\varepsilon \to 0$.

4.2. Nearly optimal control. In this subsection, we first build the nearly optimal control of the limit problem (3.9)–(3.11).

THEOREM 4.5. *Suppose that assumptions* (H1)–(H4) *hold,* $p \geq \max\{4\gamma_1, 4\gamma_2, \gamma_3\}$*,* $\{m^n(\cdot)\}_{n\geq 1} \subseteq \mathcal{R}^0$ *is a sequence of the admissible relaxed controls, and* $Y^n(\cdot)$ *denotes the solution to* (3.9) *with* $m(\cdot) = m^n(\cdot)$*. Then, the following assertions hold:*

(i) *the sequence* $\{(Y^n(\cdot), m^n(\cdot))\}_{n\geq 1}$ *is tight in* $C([0, T_0]; \mathbb{R}^{d_2}) \times \mathcal{R}(U \times [0, T_0])$*, and the limit of any weakly convergent subsequence (still indexed by* n*) satisfies* (3.9)*;*

(ii) *if* $m^n(\cdot) \Rightarrow m(\cdot)$*, then* $m(\cdot) \in \mathcal{R}^0$ *and* $J(m^n(\cdot)) \to J(m(\cdot))$ *as* $n \to \infty$*;*

(iii) *there exists an optimal control in* $\mathcal{R}^0$*.*

Proof. Similar to Theorem 4.1, (i) and (ii) can be verified by using (3.12), the weak convergence, and the martingale method. Next, we proceed to prove (iii). Since $|v^0| < \infty$, according to the definition of infimum, for any $n \geq 1$, we can choose an

$m^n(\cdot) \in \mathcal{R}^0$ such that $v^0 \le J(m^n(\cdot)) < v^0 + \frac{1}{n}$. Let $Y^n(\cdot)$ denote the solution to (3.9) corresponding to the control $m^n(\cdot)$. Then, by virtue of (i), (ii) and denoting by $(Y^*(\cdot), m^*(\cdot))$ the limit process, we have that $(Y^*(\cdot), m^*(\cdot))$ satisfies (3.9), $m^*(\cdot) \in \mathcal{R}^0$, and

$$J(m^n(\cdot)) \to J(m^*(\cdot)) \quad \text{as } n \to \infty.$$

Therefore, $J(m^*(\cdot)) = v^0$, and the desired result follows. □

Compared with the control problem of the classical SDEs, the relaxed controls can also be approximated by the ordinary controls for the distribution-dependent case. This result is collected in the following theorem.

THEOREM 4.6. *Suppose that assumptions* (H1)–(H4) *hold,* $p \ge \max\{4\gamma_1, 4\gamma_2, \gamma_3\}$, *and for a relaxed control* $m(\cdot) \in \mathcal{R}^0$, (3.9) *has a unique weak solution. Then, for any* $\delta > 0$*, there exists a piecewise constant ordinary admissible control* $u_m^\delta(\cdot)$ *for the limit problem* (3.9)–(3.11) *such that*

$$|J(u_m^\delta(\cdot)) - J(m(\cdot))| \le \delta.$$

This theorem can be established by using the same method as the case of the classical SDEs [13, 23]. So, we omit the proof. Moreover, by applying the optimal relaxed control $m^*(\cdot)$ to Theorem 4.6, we can get a δ-optimal control $u_{m^*}^\delta(\cdot)$ of the limit problem (3.9)–(3.11).

Next, the theorem below illustrates that nearly optimal control of the original problem (1.1)–(1.3) (or the revised problem with relaxed control (3.1)–(3.3)) can be obtained by simply solving the limit problem (3.9)–(3.11).

THEOREM 4.7. *Let assumptions* (H1)–(H4) *hold and* $p \ge \max\{4\gamma_1, 4\gamma_2, \gamma_3\}$. *Suppose that for any* $\delta > 0$*, there exists a feedback control* $u^\delta(\cdot)$ *with the Lipschitz continuous function* $u_0^\delta(y, \nu)$*, which is* δ*-optimal for the limit problem* (3.9)–(3.11). *Then, the revised problem with relaxed control* (3.1)–(3.3) *converges to the limit problem* (3.9)–(3.11) *in the sense of the convergence of their optimal values:*

$$\lim_{\varepsilon \to 0} v^\varepsilon = v^0.$$

Moreover, the feedback control associated with u_0^δ *is nearly optimal for the revised problem with relaxed control and the original problem in the sense:*

$$\limsup_{\varepsilon \to 0} [J^\varepsilon(u_0^\delta) - v^\varepsilon] \le \delta$$

and

$$\limsup_{\varepsilon \to 0} \left[J^\varepsilon(u_0^\delta) - \inf_{u^\varepsilon(\cdot) \in \mathcal{U}^\varepsilon} J^\varepsilon(u^\varepsilon(\cdot))\right] \le \delta,$$

respectively.

Proof. First, by Theorem 4.1 and Remark 4.4, we get

$$J^\varepsilon(u_0^\delta) \to J(u_0^\delta) \quad \text{as } \varepsilon \to 0. \tag{4.26}$$

Since $u^\delta(t) = u_0^\delta(Y(t), \mathscr{L}(Y(t)))$ is a δ-optimal control of the limit problem, we have

$$J(u_0^\delta) \le v^0 + \delta. \tag{4.27}$$

According to (4.26), (4.27), and the definition of the limit, one can get

$$v^\varepsilon \leq J^\varepsilon(u_0^\delta) \leq J(u_0^\delta) + \theta_\varepsilon \leq v^0 + \delta + \theta_\varepsilon,$$

where θ_ε is a function of ε such that $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. This together with the arbitrariness of δ implies that

$$\lim_{\varepsilon \to 0} v^\varepsilon \leq v^0. \tag{4.28}$$

On the other hand, by the definition of infimum, we can choose $m^\varepsilon(\cdot) \in \mathcal{R}^\varepsilon$ such that $v^\varepsilon \leq J^\varepsilon(m^\varepsilon(\cdot)) < v^\varepsilon + \varepsilon$. Then, it follows from Theorem 4.1 that there exists a subsequence of $m^\varepsilon(\cdot)$ (still denoted by $m^\varepsilon(\cdot)$) such that $m^\varepsilon(\cdot) \Rightarrow m(\cdot)$ and $J^\varepsilon(m^\varepsilon(\cdot)) \to J(m(\cdot))$ as $\varepsilon \to 0$. Consequently,

$$\lim_{\varepsilon \to 0} v^\varepsilon \geq \lim_{\varepsilon \to 0} J^\varepsilon(m^\varepsilon(\cdot)) = J(m(\cdot)) \geq v^0.$$

This together with (4.28) yields that

$$\lim_{\varepsilon \to 0} v^\varepsilon = v^0.$$

Moreover, using (4.26) and (4.27), we have

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} [J^\varepsilon(u_0^\delta) - v^\varepsilon] = J(u_0^\delta) - v^0 \leq \delta, \\
&\limsup_{\varepsilon \to 0} \left[J^\varepsilon(u_0^\delta) - \inf_{u^\varepsilon(\cdot) \in \mathcal{U}^\varepsilon} J^\varepsilon(u^\varepsilon(\cdot))\right] \leq \limsup_{\varepsilon \to 0} [J^\varepsilon(u_0^\delta) - v^\varepsilon] \leq \delta.
\end{aligned}$$

Thus, the proof is completed. ∎

5. Concluding remarks. In this paper, we have derived the near optimality of the optimal control problems for the singularly perturbed McKean–Vlasov systems by building the nearly optimal control of the corresponding limit problem. To obtain the limit problem, we have proved the existence of the invariant probability measure for the fast process and the weak convergence of the sequence $\{(Y^\varepsilon(\cdot), m^\varepsilon(\cdot))\}_{0<\varepsilon\leq 1}$. The results have generalized the results in [23, 24] to the McKean–Vlasov SDEs.

The coupling of the fast-slow processes is ubiquitous in practical applications, such as in manufacturing systems. Hence, an interesting topic is to treat the fully coupled systems in which the fast variable $X^\varepsilon(t)$ depends on the slow variables $Y^\varepsilon(t)$ and $\mathscr{L}(Y^\varepsilon(t))$. However, the current approaches cannot be used to deal with the corresponding systems; some novel methods are needed. This problem is challenging and deserves further investigation.

Appendix A. Proof of Proposition 3.1. We divide the proof into three steps.

Proof. Step 1*: Existence and uniqueness of the strong solution.* For any $T > 0$, under assumption (H1), b and σ are continuous and linearly growing. Consequently, (3.4) has a weak solution $(\tilde{X}, \hat{\tilde{W}})$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $\{\tilde{\mathcal{F}}_t\}_{0\leq t\leq T}$ with the initial value $\mathscr{L}(\tilde{X}(0)) = \mathscr{L}(\xi)$. For more details, readers can refer to [17].

In what follows, we establish the pathwise uniqueness of the weak solution. Let $(\hat{X}, \hat{\tilde{W}})$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $\{\hat{\mathcal{F}}_t\}_{0\leq t\leq T}$ be another weak solution to (3.4) with common initial value, i.e., $\tilde{\mathbb{P}}\{\tilde{X}(0) = \hat{X}(0)\} = 1$. Then, by the Itô formula and (H2), we obtain

$$
\begin{aligned}
|\tilde{X}(t)-\hat{X}(t)|^2 = &\int_0^t [2\langle \tilde{X}(s)-\hat{X}(s), b(\tilde{X}(s),\mathscr{L}(\tilde{X}(s))) - b(\hat{X}(s),\mathscr{L}(\hat{X}(s)))\rangle \\
&+\|\sigma(\tilde{X}(s),\mathscr{L}(\tilde{X}(s))) - \sigma(\hat{X}(s),\mathscr{L}(\hat{X}(s)))\|^2]ds \\
&+2\int_0^t \langle \tilde{X}(s)-\hat{X}(s), (\sigma(\tilde{X}(s),\mathscr{L}(\tilde{X}(s))) - \sigma(\hat{X}(s),\mathscr{L}(\hat{X}(s))))d\hat{\tilde{W}}(s)\rangle \\
\le L_1 &\int_0^t W_2(\mathscr{L}(\tilde{X}(s)),\mathscr{L}(\hat{X}(s)))^2 ds \\
&+2\int_0^t \langle \tilde{X}(s)-\hat{X}(s), (\sigma(\tilde{X}(s),\mathscr{L}(\tilde{X}(s))) - \sigma(\hat{X}(s),\mathscr{L}(\hat{X}(s))))d\hat{\tilde{W}}(s)\rangle.
\end{aligned}
$$

Taking the expectation on both sides and using the Grönwall inequality, one obtains

$$\tilde{\mathbb{E}}|\tilde{X}(t)-\hat{X}(t)|^2 = 0.$$

This together with the continuity of $\tilde{X}(\cdot)$ and $\hat{X}(\cdot)$ implies that the pathwise uniqueness holds. Let $\tilde{\mu}_t = \mathscr{L}(\tilde{X}(t))$, and consider the classical SDE

$$dX(t) = b(X(t),\tilde{\mu}_t)dt + \sigma(X(t),\tilde{\mu}_t)d\hat{W}^1(t). \tag{A.1}$$

For initial value ξ, we get that $(\tilde{X},\hat{\tilde{W}})$, $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, $\{\tilde{\mathcal{F}}_t\}_{0\le t\le T}$ is a weak solution to (A.1). Using the same technique as the above proof of the pathwise uniqueness and the classical Yamada–Watanabe principle [19], (A.1) has a unique strong solution $(X(t))_{0\le t\le T}$ satisfying $X(0)=\xi$. Thus, by the modified Yamada–Watanabe principle [15], the arbitrariness of T implies that (3.4) has a unique strong solution $(X(t))_{t\ge 0}$.

Step 2*: **P**th moment estimate* (3.5). For any $x\in\mathbb{R}^{d_1}$ and $\mu\in\mathcal{P}_2(\mathbb{R}^{d_1})$, assumptions (H1), (H2), and the Young inequality imply that

$$2\langle x, b(x,\mu)\rangle + (p-1)\|\sigma(x,\mu)\|^2 \le L_0 - (L_2-\theta)|x|^2 + (L_1+\theta)[\mu]_2, \tag{A.2}$$

where $\theta = \frac{L_2-L_1}{4} > 0$ and L_0 is a positive constant. By the Itô formula and (A.2), one can get

$$
\begin{aligned}
&|X(t)|^p e^{\frac{p}{2}(L_2-L_1-3\theta)t} \\
&\le |\xi|^p + \frac{p}{2}(L_2-L_1-2\theta)\int_0^t e^{\frac{p}{2}(L_2-L_1-3\theta)s}|X(s)|^p ds + C\int_0^t e^{\frac{p}{2}(L_2-L_1-3\theta)s}ds \\
&\quad +\frac{p}{2}\int_0^t e^{\frac{p}{2}(L_2-L_1-3\theta)s}|X(s)|^{p-2}[(L_1+\theta)W_2(\mathscr{L}(X(s)),\delta_0)^2 - (L_2-\theta)|X(s)|^2]ds \\
&\quad +p\int_0^t e^{\frac{p}{2}(L_2-L_1-3\theta)s}|X(s)|^{p-2}\langle X(s),\sigma(X(s),\mathscr{L}(X(s)))d\hat{W}^1(s)\rangle.
\end{aligned}
$$

Taking the expectation and using the Hölder inequality, we have

$$\mathbb{E}|X(t)|^p \le \mathbb{E}|\xi|^p\cdot e^{-\frac{p}{2}(L_2-L_1-3\theta)t} + C\int_0^t e^{-\frac{p}{2}(L_2-L_1-3\theta)(t-s)}ds \le C,$$

where constant C is independent of t. Therefore, the desired conclusion is acquired.

Step 3*: Existence and uniqueness of the invariant probability measure.* At this step, for the sake of completeness, we will only give a brief proof of the existence and uniqueness of the invariant probability measure for (3.4). For more details, readers can refer to [39, Theorem 3.1]. For any $\mu_0, \mu_0' \in \mathcal{P}_p(\mathbb{R}^{d_1})$, let $(X(t))_{t\geq 0}$ and $(X'(t))_{t\geq 0}$ be two solutions to (3.4) such that $\mathscr{L}(X(0)) = \mu_0$, $\mathscr{L}(X'(0)) = \mu_0'$, and $W_p(\mu_0, \mu_0')^p = \mathbb{E}|X(0) - X'(0)|^p$. Set $P_t^*\mu_0 = \mathscr{L}(X(t))$ and $P_t^*\mu_0' = \mathscr{L}(X'(t))$ for any $t \geq 0$. Then, by the Itô formula, assumption (H2), and (3.4), it follows that

$$
\begin{aligned}
&|X(t) - X'(t)|^p e^{\frac{p}{2}(L_2 - L_1)t} \\
&\leq |X(0) - X'(0)|^p - \frac{p}{2} L_1 \int_0^t e^{\frac{p}{2}(L_2 - L_1)s} |X(s) - X'(s)|^p ds \\
&\quad + \frac{p}{2} L_1 \int_0^t e^{\frac{p}{2}(L_2 - L_1)s} |X(s) - X'(s)|^{p-2} W_2(\mathscr{L}(X(s)), \mathscr{L}(X'(s)))^2 ds \\
&\quad + p \int_0^t e^{\frac{p}{2}(L_2 - L_1)s} |X(s) - X'(s)|^{p-2} \langle X(s) - X'(s), \\
&\qquad\qquad (\sigma(X(s), \mathscr{L}(X(s))) - \sigma(X'(s), \mathscr{L}(X'(s)))) d\hat{W}^1(s) \rangle.
\end{aligned}
$$

Taking the expectation and using the definition of L^p-Wasserstein distance, one gets

$$
W_p(P_t^*\mu_0, P_t^*\mu_0')^p \leq \mathbb{E}|X(t) - X'(t)|^p \leq W_p(\mu_0, \mu_0')^p e^{-\frac{p}{2}(L_2 - L_1)t} \quad \forall t \geq 0. \tag{A.3}
$$

Then, by (A.3) and the relation

$$
P_t^*(P_s^*\delta_0) = P_{t+s}^*\delta_0 \quad \forall s, t \geq 0, \tag{A.4}
$$

we have that $\{P_t^*\delta_0\}_{t\geq 0}$ is a Cauchy sequence in $\mathcal{P}_p(\mathbb{R}^{d_1})$. Therefore, there exists a $\mu \in \mathcal{P}_p(\mathbb{R}^{d_1})$ such that

$$
\lim_{t\to\infty} W_p(P_t^*\delta_0, \mu) = 0. \tag{A.5}
$$

Moreover, using (A.3)–(A.5), we can prove that μ is an invariant probability measure, and for any initial distribution $\mu_0 \in \mathcal{P}_p(\mathbb{R}^{d_1})$,

$$
W_p(P_t^*\mu_0, \mu)^p = W_p(P_t^*\mu_0, P_t^*\mu)^p \leq W_p(\mu_0, \mu)^p e^{-\frac{p}{2}(L_2 - L_1)t} \quad \forall t \geq 0.
$$

The proof is completed. ∎

Appendix B. Proof of Proposition 3.3. We divide the proof into two steps.

Proof. Step 1*: Existence and uniqueness of the strong solution.* For any $m^\varepsilon(\cdot) \in \mathcal{R}^\varepsilon$, applying assumption (H3), we have that for any $t \geq 0$, $\omega \in \Omega$, $u \in \mathbb{R}^r$, $y_1, y_2 \in \mathbb{R}^{d_2}$, and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2})$,

$$
\begin{aligned}
&|f(X^\varepsilon(t), \mu_t^\varepsilon, y_1, \nu_1, u) - f(X^\varepsilon(t), \mu_t^\varepsilon, y_2, \nu_2, u)| \\
&\vee \|g(X^\varepsilon(t), \mu_t^\varepsilon, y_1, \nu_1) - g(X^\varepsilon(t), \mu_t^\varepsilon, y_2, \nu_2)\| \\
&\leq 2L(|y_1 - y_2| + W_2(\nu_1, \nu_2)),
\end{aligned}
$$

where $\mu_t^\varepsilon = \mathscr{L}(X^\varepsilon(t))$. Moreover, assumption (H4) and Remark 3.2 give that for any $T > 0$,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \int_U |f(X^\varepsilon(t), \mu_t^\varepsilon, 0, \delta_0, u)|^4 m_t^\varepsilon(du) dt\Big] \leq CK^4 \int_0^T (1 + \mathbb{E}|X^\varepsilon(t)|^{4\gamma_1}) dt \leq C_T, \\
&\mathbb{E}\Big[\int_0^T \|g(X^\varepsilon(t), \mu_t^\varepsilon, 0, \delta_0)\|^4 dt\Big] \leq CK^4 \int_0^T (1 + \mathbb{E}|X^\varepsilon(t)|^{4\gamma_2}) dt \leq C_T.
\end{aligned}
$$

These, together with the existence and uniqueness of the solution to McKean–Vlasov SDEs, imply that the slow-varying equation in (3.1) has a unique strong solution on $[0,T]$; see [10, Theorem 3.3] for more details. Consequently, the arbitrariness of T yields the desired result.

Step 2*: Moment estimate* (3.6)*.* Using the Hölder inequality, we compute

$$\begin{aligned}|Y^\varepsilon(t)|^4 \le 27|\zeta|^4 + 27T^3\int_0^t\int_U |f(X^\varepsilon(s),\mu_s^\varepsilon,Y^\varepsilon(s),\nu_s^\varepsilon,u)|^4 m_s^\varepsilon(du)ds\\ +27\Big|\int_0^t g(X^\varepsilon(s),\mu_s^\varepsilon,Y^\varepsilon(s),\nu_s^\varepsilon)dW^2(s)\Big|^4,\end{aligned}$$

where $\nu_s^\varepsilon = \mathscr{L}(Y^\varepsilon(s))$. For any $t\in[0,T]$, by assumptions (H3), (H4), the Burkholder–Davis–Gundy inequality, and Remark 3.2, one can calculate that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le s\le t}|Y^\varepsilon(s)|^4\Big]\\ &\le C\mathbb{E}|\zeta|^4 + CL^4T(T^2+1)\mathbb{E}\Big[\int_0^t\big(|Y^\varepsilon(s)|^4 + W_2(\nu_s^\varepsilon,\delta_0)^4\big)ds\Big]\\ &\quad + CK^4T(T^2+1)\mathbb{E}\Big[\int_0^T\big(1+|X^\varepsilon(s)|^{4\gamma_1}+|X^\varepsilon(s)|^{4\gamma_2}+W_2(\mu_s^\varepsilon,\delta_0)^4\big)ds\Big]\\ &\le C_T(1+\mathbb{E}|\zeta|^4) + C_T\int_0^t\mathbb{E}\Big[\sup_{0\le r\le s}|Y^\varepsilon(r)|^4\Big]ds.\end{aligned}$$

Furthermore, by the Grönwall inequality, the required assertion follows. ∎

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